Integral Reduction from Elliptic and Hyperelliptic Curves

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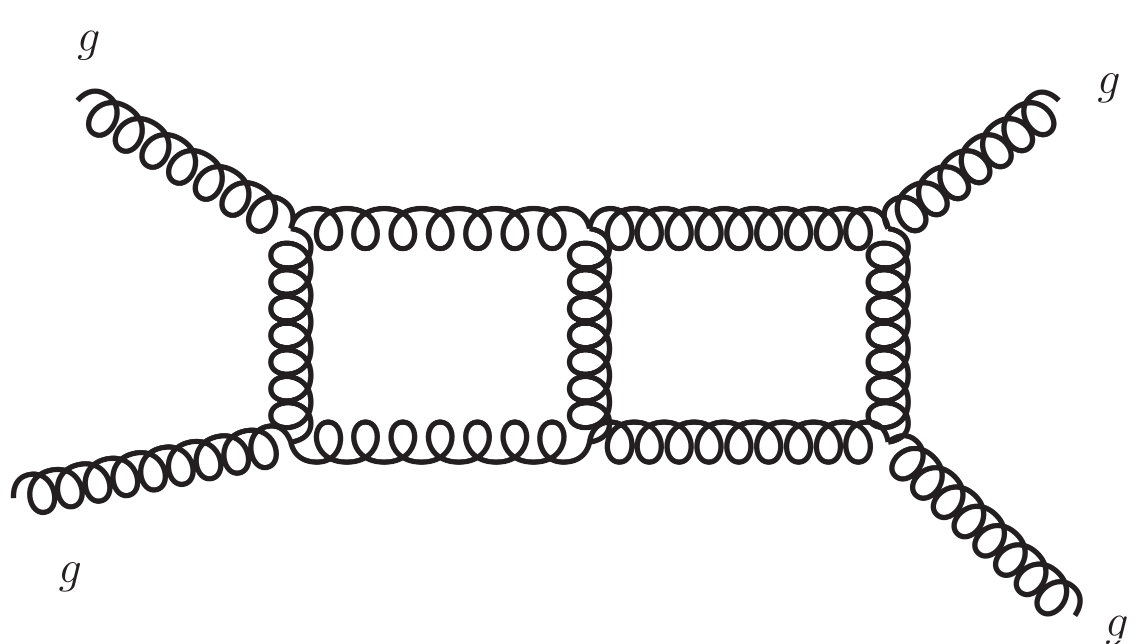
Loopfest 2015

IBP relations

also see Alexander Smirnov's talk

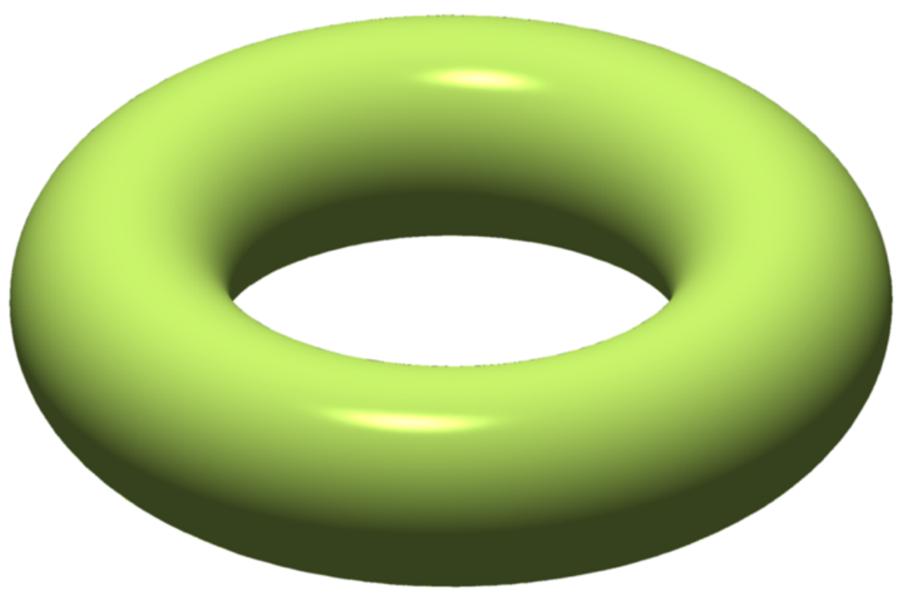
$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1 \ldots D_k} \right) = 0$$

Reduce the set of Feynman integrals to master integrals
difficult to find IBP relations for massive/multi-leg high loop diagrams

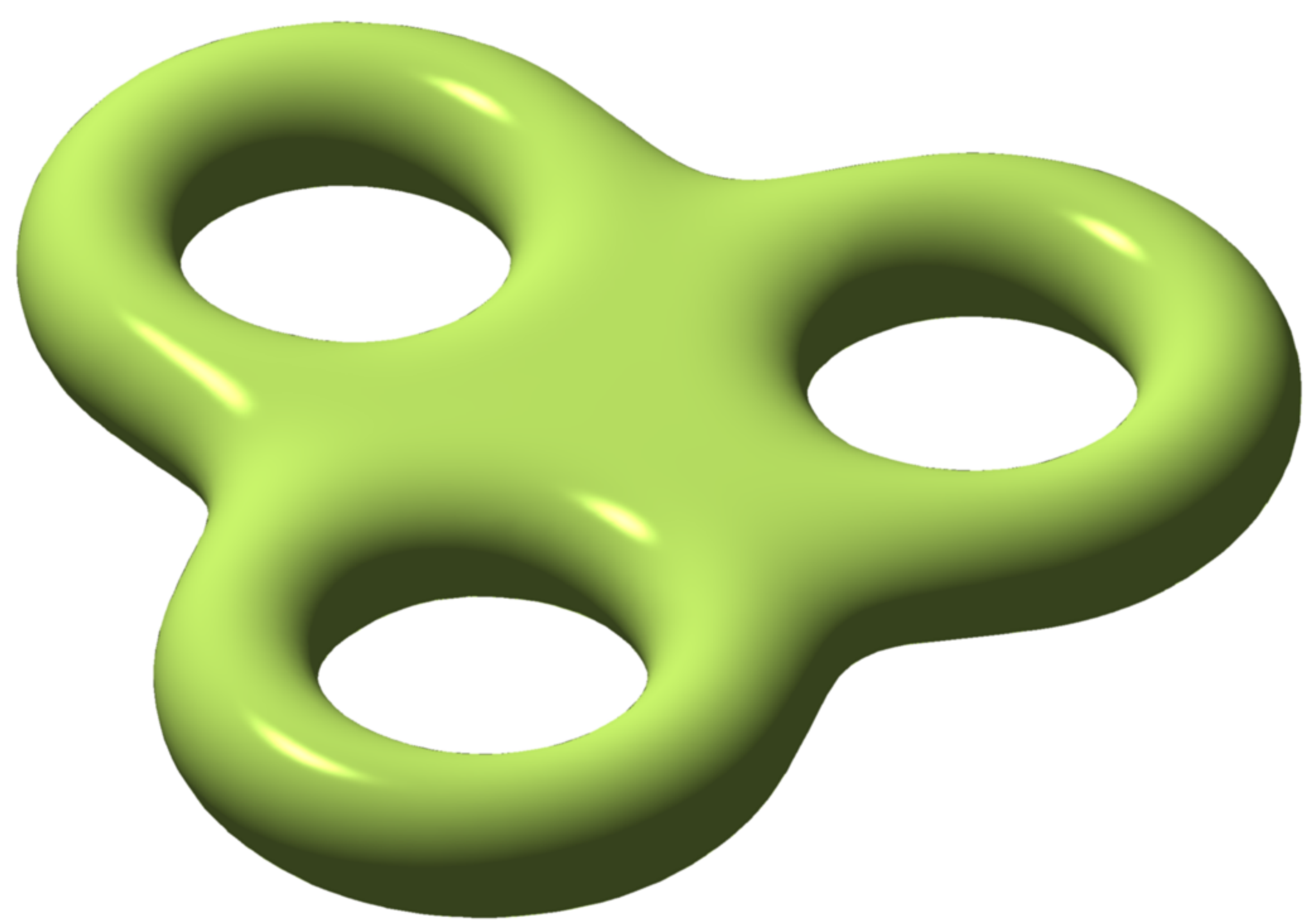


IBP from geometric approach?

Overview



- IBP relations from the calculus of elliptic curves
- IBP relations from the calculus of hyperelliptic curves



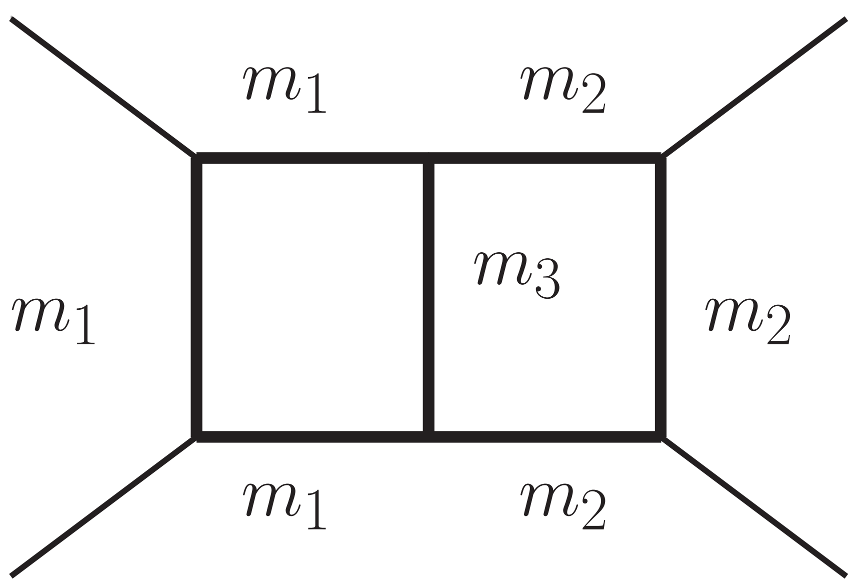
Based on
Mads Sogaard and YZ, 1412.5577
Alessandro Georgoudis and YZ, 1507.xxxxx

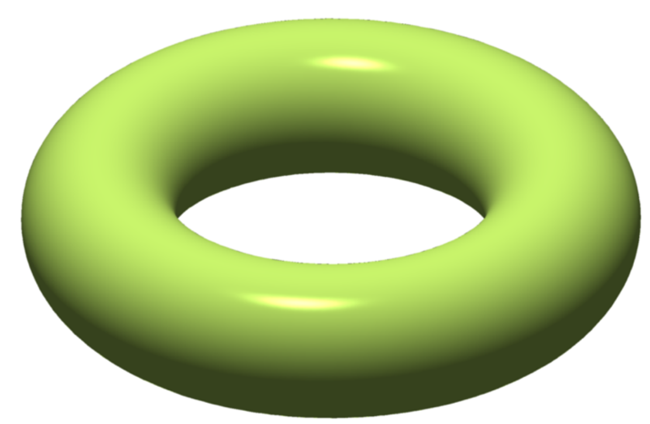
For a D-dimensional L-loop Feynman integral, (1) with $DL-1$ propagates and (2) with smooth unitarity cut, the "on-shell" parts of IBPs correspond to exact meromorphic one-forms on an algebraic curve.

Unitarity and Algebraic Curves (4D)

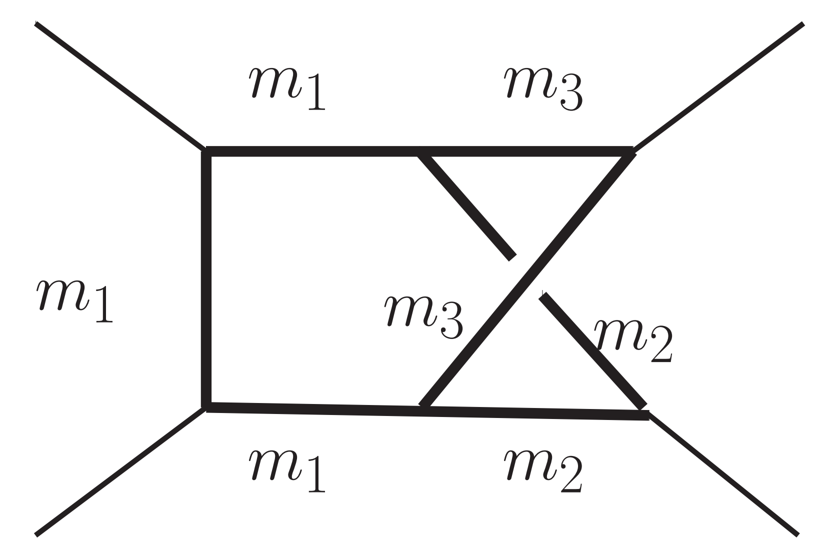
$D_1 = \ldots D_k = 0$

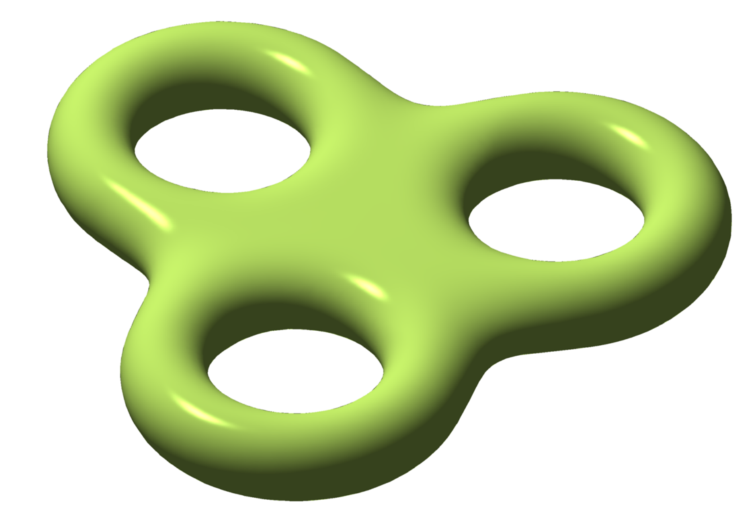
J.Hauenstein, R. Huang, D. Mehta, YZ   1408.3355



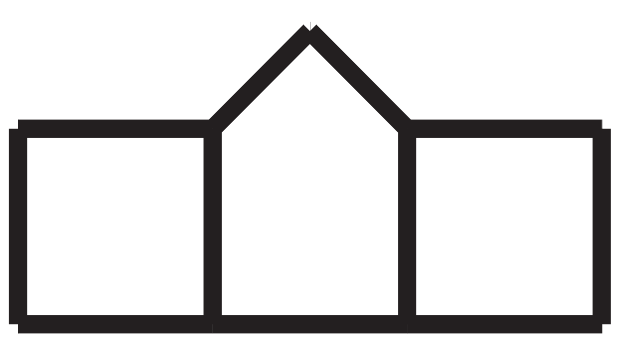


genus=1

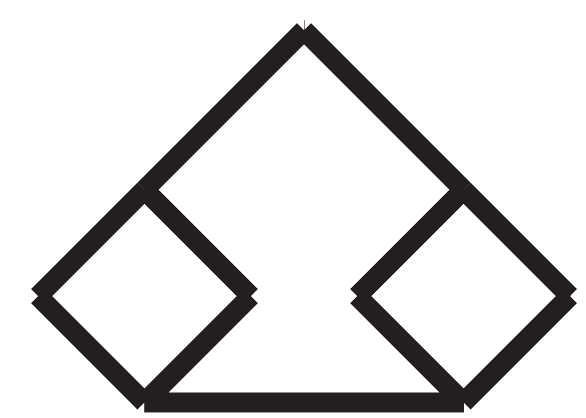




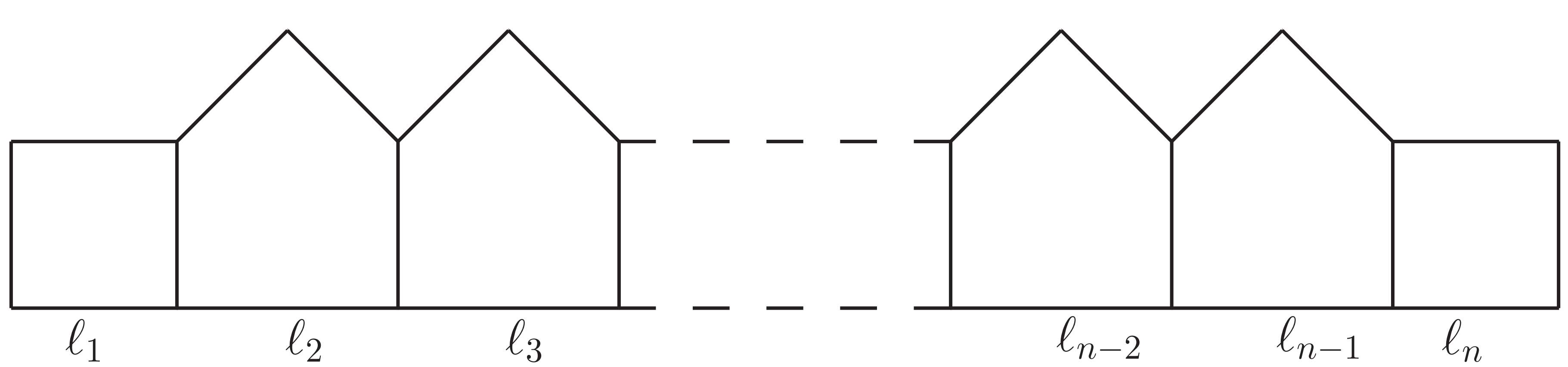
genus=3



genus=5



genus=13



$$g = (n-2)2^{n-1} + 1$$

Riemann-Hurwitz formula

genus is odd,
if smooth

Unitarity and IBP relations

also see Kasper Larsen's talk

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots l_L)}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = c_i I_i + \ldots$$

D. Kosower, K.Larsen 1108.1180

Integrals with fewer propagators

Unitarity $$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots l_L)}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} \rightarrow \oint_{V:\{D_1=\ldots=D_k=0\}} \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots l_L)}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}}$$

$$= \oint_{\text{poles,fundamental cycles of } V} \omega$$

determine all the c_i's

ω is a meromorphic differential form on V. The contours are around the poles of ω and the fundamental cycles of V.

$$\oint_{\text{poles,fundamental cycles of } V} \omega = 0 \iff \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots l_L)}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = 0 + \ldots$$

Integrable globally;
Exact meromorphic form $\iff$ One-shell part of IBP

Unitarity and IBP relations

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D. Kosower, K.Larsen 1108.1180

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \dots l_L)}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = c_i I_i + \dots$$

Integrals with fewer propagators

Unitarity
$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \dots l_L)}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} \rightarrow \oint_{V:\{D_1=\ldots=D_k=0\}} \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \dots l_L)}{D_1^{\alpha_1} \dots D_k^{\alpha_k}}$$
$$= \oint_{\text{poles,fundamental cycles of } V} \omega$$

determine all the c_i's

ω is a meromorphic differential form on V. The contours are around the poles of ω and the fundamental cycles of V.

$$\oint_{\text{poles,fundamental cycles of } V} \omega = 0 \quad \Longleftrightarrow \quad \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \dots l_L)}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = 0 + \dots$$

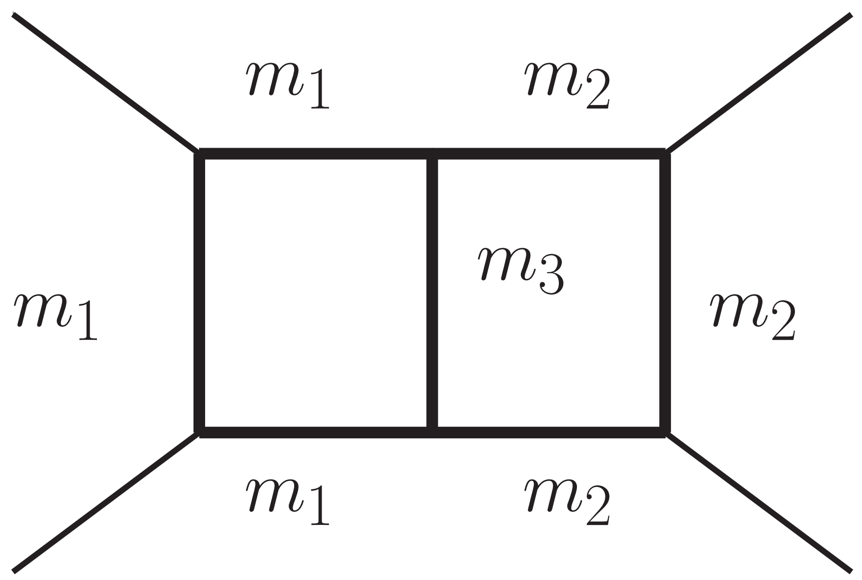
Integrable globally;
Exact meromorphic form

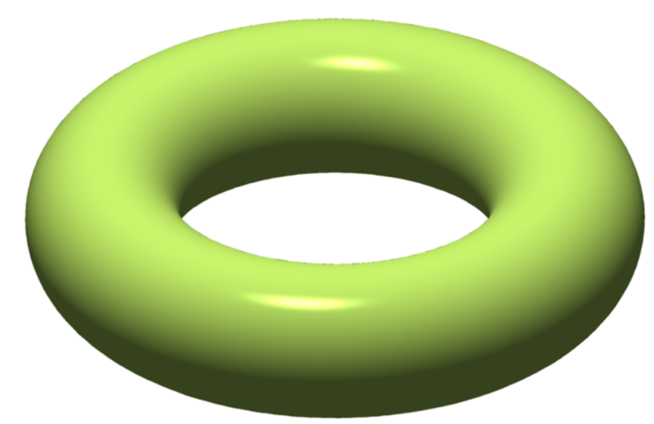
$\Longleftrightarrow$ One-shell part of IBP

Easy to find

Elliptic example

Mads Sogaard and YZ, 1412.5577





$$\ell_1^\mu = a_1 p_1^\mu + a_2 p_2^\mu + a_3 \frac{s_{12}\langle 1|\gamma^\mu|2]}{2\langle 14\rangle[42]} + a_4 \frac{s_{12}\langle 2|\gamma^\mu|1]}{2\langle 24\rangle[41]},$$
$$\ell_2^\mu = b_1 p_3^\mu + b_2 p_4^\mu + b_3 \frac{s_{12}\langle 3|\gamma^\mu|4]}{2\langle 31\rangle[14]} + b_4 \frac{s_{12}\langle 4|\gamma^\mu|3]}{2\langle 41\rangle[13]}.$$

$$a_1 = 1, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 1,$$
$$a_3 = \frac{C_1}{a_4}, \quad b_3 = \frac{C_2}{b_4}$$

$$\boxed{A(a_4)b_4^2 + B(a_4)b_4 + C(a_4) = 0}$$

Elliptic curve

$$b_4 = \frac{-B(a_4) + \sqrt{\Delta(\alpha)}}{2A(a_4)}$$

degree-4

$$\int \frac{d^4 l_1}{i\pi^2}\frac{d^4 l_2}{i\pi^2}\frac{N(l_1,l_2)}{D_1 \ldots D_7} \to \oint \frac{da_4}{\sqrt{\Delta(a_4)}} N(a_i,b_j)$$

Holomorphic 1-form
of Elliptic curve

Goal: To find $N(a_i, b_j)$ such that $\frac{N(a_i,b_j)da_4}{\sqrt{\Delta(a_4)}}$ is exact

Exact 1-form can be found by simple differential operation on elliptic curve

$$da_4 = (2A(a_4)b_4 + B(a_4))\frac{da_4}{\sqrt{\Delta(a_4)}}$$
$$db_4 = (-A'(a_4)b_4^2 - B'(a_4)b_4 - C'(a_4))\frac{da_4}{\sqrt{\Delta(a_4)}}$$

All on-shell IBPs found,
5 master integrals

Just use rational functions, elliptic function is NOT needed!

Exact 1-form can be found by simple differential operation on elliptic curve

Desired N

$$da_4 = (2A(a_4)b_4 + B(a_4))\frac{da_4}{\sqrt{\Delta(a_4)}}$$

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All on-shell IBPs found,
5 master integrals

Just use rational functions, elliptic function is NOT needed!

Works as well for integrals with doubled propagators

$$\int \frac{d^4 l_1}{i\pi^2}\frac{d^4 l_2}{i\pi^2}\frac{N(l_1,l_2)}{D_1^2 D_2 D_3 D_4 D_5 D_6 D_7} \to \oint \frac{F(a_4)da_4}{\Delta(a_4)^{3/2}} N(a_i,b_j) \qquad \langle \Delta(a_4), \Delta'(a_4)\rangle = 1$$

$$F(a_4) = f_1(a_4)\Delta'(a_4) + f_2(a_4)\Delta(a_4)$$

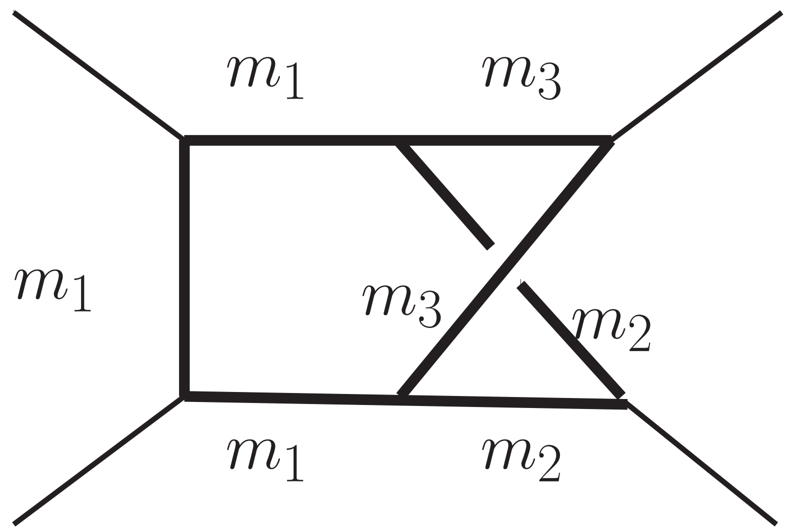
$$d\left(\frac{-2f_1(a_4)}{\sqrt{\Delta(a_4)}}\right) = \frac{f_1(a_4)\Delta'(a_4)}{\Delta(a_4)^{3/2}}da_4 - 2\frac{f_1'(a_4)}{\sqrt{\Delta(a_4)}}da_4$$

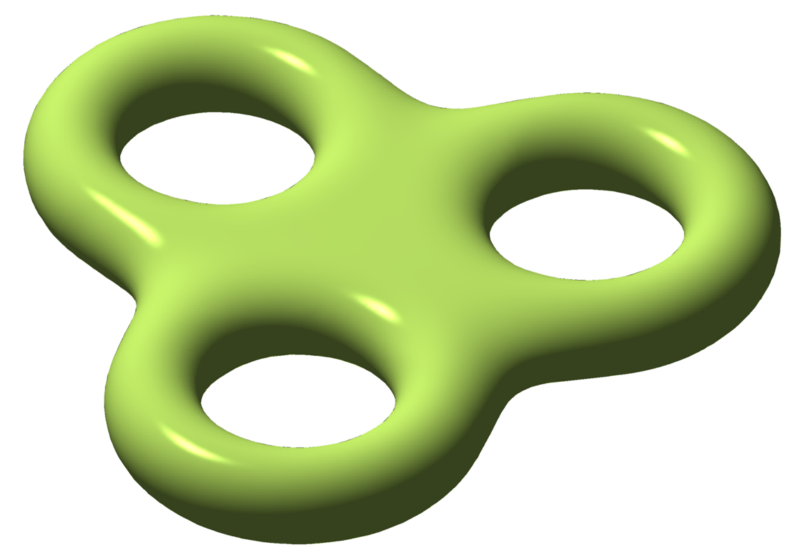
Exact meromorphic form

$$\int \frac{d^4 l_1}{i\pi^2}\frac{d^4 l_2}{i\pi^2}\frac{N(l_1,l_2)}{D_1^2 D_2 D_3 D_4 D_5 D_6 D_7} \to \oint \frac{da_4}{\sqrt{\Delta(a_4)}} N(a_i,b_j)(2f_1'(a_4) + f_2(a_4))$$ doubled propagator removed

very easy to find exact 1-form to reduce a given target integral

Hyperelliptic example: genus=3





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Holomorphic 1-form

$$\int \frac{d^4 l_1}{i\pi^2}\frac{d^4 l_2}{i\pi^2}\frac{N(l_1,l_2)}{D_1\ldots D_k} \to \oint \frac{a_4 da_4}{\sqrt{\Delta(a_4)}} N(a_i,b_j)$$

$$A(a_4)b_3^2 + B(a_4)b_3 + C(a_4) = 0$$

Hyperelliptic curve

$$b_3 = \frac{-B(a_4)+\sqrt{\Delta(a_4)}}{2A(a_4)}$$

degree-8

Just use rational functions, hyperelliptic function is NOT needed!

$$I_{\text{massive xbox}}[2a_4^2 b_3 s^5 + 2a_4 b_3 s^4(s+t) + a_4 s(s+t)\left(m_1^2\left(s^2+2st+2t^2\right) + s\left(m_2^2(-s)+m_3^2 s + t(s+t)\right)\right)$$
$$+a_3 m_1^2 st(s+t)^2 + a_4^3 s^4(s+t) + a_4^2 s^3(s+t)(s+2t) + 2b_3 m_1^2 s^2(s+t)^2 + m_1^2 t(s+t)^2(s+2t)] = 0 + \ldots$$

… …

All on-shell IBPs found,
7 master integrals

Numerically consistent with FIRE

Summary

- Geometric meaning for the on-shell part of IBP relations: exact meromorphic forms
- Works for both two-loop planar and non-planar diagrams
- No explicit form of elliptic/hyperelliptic function is needed

Future directions

- Two-loop D=4-2ε unitarity

  Kasper Larsen and YZ, appear soon
- Integral counting from Riemann-Roch theorem

  Alessandro and YZ, 1507.xxxxx
- Unitarity cut with singular point
- Algebraic surface cases