Integral Reduction from Elliptic and Hyperelliptic Curves

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IBP relations

\[ \int \frac{dl_1^D}{i \pi^{D/2}} \cdots \int \frac{dl_L^D}{i \pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \left( \frac{v_i^\mu}{D_1 \cdots D_k} \right) = 0 \]

also see Alexander Smirnov’s talk

Reduce the set of Feynman integrals to \textit{master integrals} difficult to find IBP relations for massive/multi-leg high loop diagrams

IBP from \textit{geometric} approach?
Overview

- IBP relations from the calculus of elliptic curves
- IBP relations from the calculus of hyperelliptic curves

Based on
Mads Sogaard and YZ, 1412.5577
Alessandro Georgoudis and YZ, 1507.xxxxx

For a $D$-dimensional $L$-loop Feynman integral, (1) with $DL - 1$ propagates and (2) with smooth unitarity cut, the “on-shell” parts of IBPs correspond to exact meromorphic one-forms on an algebraic curve.
Unitarity and Algebraic Curves (4D)

\[ D_1 = \ldots D_k = 0 \]

\[ \text{genus} = 1 \]

\[ \text{genus} = 3 \]

\[ \text{genus} = 5 \]

\[ \text{genus} = 13 \]

\[ g = (n - 2)2^{n-1} + 1 \]

Riemann-Hurwitz formula
Unitarity and IBP relations

\[ \int \frac{d^D l_1}{i \pi^{D/2}} \cdots \frac{d^D l_L}{i \pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} = c_i I_i + \ldots \]

Integrals with fewer propagators

Unitarity

\[ \int \frac{d^D l_1}{i \pi^{D/2}} \cdots \frac{d^D l_L}{i \pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} \rightarrow \int_{V: \{D_1 = \ldots = D_k = 0\}} \frac{d^D l_1}{i \pi^{D/2}} \cdots \frac{d^D l_L}{i \pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} = \int_{\text{poles, fundamental cycles of } V} \omega \text{ determine all the } c_i \text{'s} \]

\( \omega \) is a meromorphic differential form on \( V \). The contours are around the poles of \( \omega \) and the fundamental cycles of \( V \).

\[ \int_{\text{poles, fundamental cycles of } V} \omega = 0 \quad \iff \quad \int \frac{d^D l_1}{i \pi^{D/2}} \cdots \frac{d^D l_L}{i \pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} = 0 + \ldots \]

Integrable globally;
Exact meromorphic form

\[ \iff \quad \text{One-shell part of IBP} \]

also see Kasper Larsen’s talk

D. Kosower, K. Larsen 1108.1180
Unitarity and IBP relations

\( \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} = c_i I_i + \ldots \)

Integrals with fewer propagators

Unitarity

\[ \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} \rightarrow \int_{V: \{D_1 = \ldots = D_k = 0\}} \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} \]

\[ = \int_{\text{poles, fundamental cycles of } V} \omega \]

\( \omega \) is a meromorphic differential form on \( V \). The contours are around the poles of \( \omega \) and the fundamental cycles of \( V \).

\[ \int_{\text{poles, fundamental cycles of } V} \omega = 0 \quad \iff \quad \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \ldots, l_L)}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}} = 0 + \ldots \]

Integrable globally; Exact meromorphic form

\[ \iff \quad \text{One-shell part of IBP} \]

Easy to find

also see Kasper Larsen’s talk

D. Kosower, K. Larsen 1108.1180
Elliptic example

$$\ell_1^\mu = a_1 p_1^\mu + a_2 p_2^\mu + a_3 \frac{s_{12} \langle 1 | \gamma^\mu | 2 \rangle}{2 \langle 14 | 42 \rangle} + a_4 \frac{s_{12} \langle 2 | \gamma^\mu | 1 \rangle}{2 \langle 24 | 41 \rangle},$$

$$\ell_2^\mu = b_1 p_3^\mu + b_2 p_4^\mu + b_3 \frac{s_{12} \langle 3 | \gamma^\mu | 4 \rangle}{2 \langle 31 | 14 \rangle} + b_4 \frac{s_{12} \langle 4 | \gamma^\mu | 3 \rangle}{2 \langle 41 | 13 \rangle}.$$

$$a_1 = 1, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 1,$$

$$a_3 = \frac{C_1}{a_4}, \quad b_3 = \frac{C_2}{b_4}$$

$$\int \frac{d^{4}l_1}{i\pi^2} \frac{d^{4}l_2}{i\pi^2} \frac{N(l_1, l_2)}{D_1 \ldots D_7} \to \oint \frac{da_4}{\sqrt{\Delta(a_4)}} N(a_i, b_j)$$

Goal: To find $N(a_i, b_j)$ such that $\frac{N(a_i, b_j) da_4}{\sqrt{\Delta(a_4)}}$ is exact
Exact 1-form can be found by simple differential operation on elliptic curve

\[ da_4 = (2A(a_4)b_4 + B(a_4)) \frac{da_4}{\sqrt{\Delta(a_4)}} \]
\[ db_4 = (-A'(a_4)b_4^2 - B'(a_4)b_4 - C''(a_4)) \frac{da_4}{\sqrt{\Delta(a_4)}} \]

All on-shell IBPs found, 5 master integrals

Just use rational functions, elliptic function is NOT needed!
Exact 1-form can be found by **simple** differential operation on elliptic curve

\[
\begin{align*}
da_4 &= (2A(a_4)b_4 + B(a_4)) \frac{da_4}{\sqrt{\Delta(a_4)}} \\
db_4 &= (-A'(a_4)b_4^2 - B'(a_4)b_4 - C'(a_4)) \frac{da_4}{\sqrt{\Delta(a_4)}}
\end{align*}
\]

All on-shell IBPs found, 5 master integrals

Just use rational functions, elliptic function is NOT needed!
Exact 1-form can be found by **simple** differential operation on elliptic curve

\[ da_4 = \frac{(2A(a_4)b_4 + B(a_4))}{\sqrt{\Delta(a_4)}} da_4 \]

\[ db_4 = \frac{(-A'(a_4)b_4^2 - B'(a_4)b_4 - C''(a_4))}{\sqrt{\Delta(a_4)}} da_4 \]

All on-shell IBPs found, 5 master integrals

Just use rational functions, elliptic function is **NOT** needed!

Works as well for integrals with doubled propagators

\[
\int \frac{d^4 l_1}{i\pi^2} \frac{d^4 l_2}{i\pi^2} \frac{N(l_1, l_2)}{D_1^2D_2D_3D_4D_5D_6D_7} \to \int \frac{F(a_4)da_4}{\Delta(a_4)^{3/2}} N(a_i, b_j) \quad \langle \Delta(a_4), \Delta'(a_4) \rangle = 1
\]

\[
F(a_4) = f_1(a_4)\Delta'(a_4) + f_2(a_4)\Delta(a_4)
\]

\[
\int \frac{d^4 l_1}{i\pi^2} \frac{d^4 l_2}{i\pi^2} \frac{N(l_1, l_2)}{D_1^2D_2D_3D_4D_5D_6D_7} \to \int \frac{da_4}{\sqrt{\Delta(a_4)}} N(a_i, b_j)(2f'_1(a_4) + f_2(a_4)) \quad \text{doubled propagator removed}
\]

very easy to find exact 1-form to reduce a given target integral
Hyperelliptic example: genus=3

\[ \ell_1^\mu = a_1 p_1^\mu + a_2 p_2^\mu + a_3 s_{12}(1|\gamma^\mu|2) + a_4 s_{12}(2|\gamma^\mu|1) - \frac{1}{2(14)[42]}, \]

\[ \ell_2^\mu = b_1 p_3^\mu + b_2 p_4^\mu + b_3 s_{12}(3|\gamma^\mu|4) + b_4 s_{12}(4|\gamma^\mu|3) - \frac{1}{2(31)[14]} + \frac{1}{2(41)[13]} . \]

Just use rational functions, hyperelliptic function is NOT needed!

\[ \int \frac{d^4l_1 d^4l_2}{i\pi^2 i\pi^2} \frac{N(l_1, l_2)}{D_1 \ldots D_k} \rightarrow \int \frac{a_4 da_4}{\sqrt{\Delta(a_4)}} N(a_i, b_j) \]

\[ A(a_4)b_3^2 + B(a_4)b_3 + C(a_4) = 0 \]

\[ b_3 = \frac{-B(a_4) + \sqrt{\Delta(a_4)}}{2A(a_4)} \]

degree-8

\[ I_{\text{massive \, box}}[2a_4b_3 s^5 + 2a_4b_3 s^4(s + t) + a_4 s(s + t) \left( m_1^2 \left( s^2 + 2st + 2t^2 \right) + s \left( m_2^2(-s) + m_3^2 s + t(s + t) \right) \right) + a_3 m_1^2 st(s + t)^2 + a_4^3 s^4(s + t) + a_4^2 s^3(s + t)(s + 2t) + 2b_3 m_1^2 s^2(s + t)^2 + m_1^2 t(s + t)^2(s + 2t)] = 0 + \ldots \]

... ...

All on-shell IBPs found, 7 master integrals

Numerically consistent with FIRE
Summary

- Geometric meaning for the on-shell part of IBP relations: exact meromorphic forms
- Works for both two-loop planar and non-planar diagrams
- No explicit form of elliptic/hyperelliptic function is needed

Future directions

- Two-loop D=4-2\epsilon unitarity
  Kasper Larsen and YZ, appear soon
- Integral counting from Riemann-Roch theorem
  Alessandro and YZ, 1507.xxxxx
- Unitarity cut with singular point
- Algebraic surface cases