Evaluating Feynman integrals by uniformly transcendental differential equations

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Based on collaboration with Johannes Henn, Bernhard Mistlberger and Alexander Smirnov

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More on the method of differential equations in other talks [L. Tancredi, O. Gituliar, B. Mistlberger, R. Schabinger, C. Papadopoulos, A. von Manteuffel, C. Wever]

[A.V. Kotikov'91, E. Remiddi'97, T. Gehrmann & E. Remiddi'00, J. Henn'13]

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A lot of applications [J.M. Henn, A.V. Smirnov, V.A. Smirnov, K. Melnikov, F. Caola, R. Bonciani, V. Del Duca, H. Frellesvig, F. Moriello, M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi, T. Gehrmann, A. von Manteuffel, E. Weihs, F. Dulat, B. Mistlberger, R. N. Lee,...]

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- Solve DE

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DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x)\,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

Henn (2013): turn to a new basis where DE take the form

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In the differential form,

$$d\, f(\epsilon, x) = \epsilon\, (d\, \tilde{A}(x))\, f(x, \epsilon)\,,$$

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$$\tilde{A} = \sum_k \tilde{A}_{\alpha_k}\, \log(\alpha_k)\,.$$

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and $\tilde{A}_{\alpha_k}$ are *constant* matrices. The arguments of the logarithms α_i (*letters*) are functions of x. Elements of such basis turn out to be uniformly transcendental (UT).

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Let us call it *epsilon form*.

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$$f'(\epsilon, x) = \epsilon \sum_k \frac{a_k}{x - x^{(k)}} f(\epsilon, x) \,.$$

where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices a_k are independent of x and ϵ.

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For example, if $x_k = 0, -1, 1$ then results for elements of such a basis are expressed in terms of HPL.

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- An approach using Magnus and Dyson series expansion [M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi'14]

- A part of the procedure is algorithmically described in [T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs'14]

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- Constructing UT elements of the basis at the level of integrand [Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz and J. Trnka'14]

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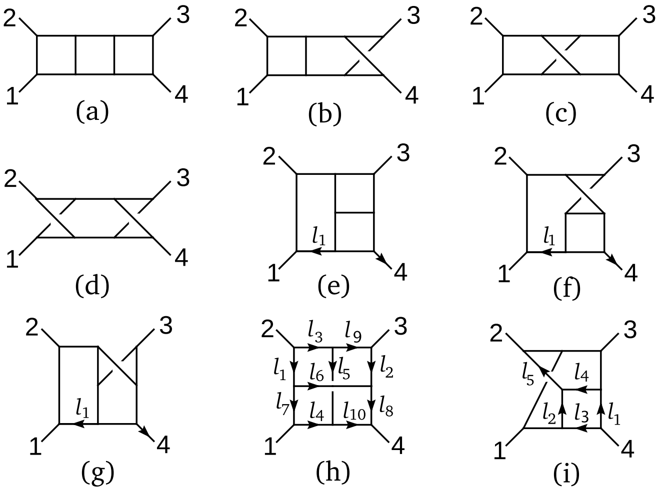
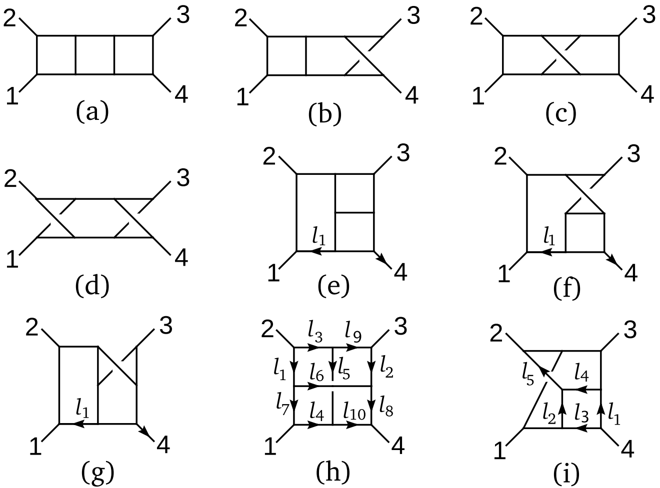
- In the case of one variables was algorithmically described [R.N. Lee'14]


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)



Motivation: three-loop amplitudes of $N = 8$ supergravity and $N = 4$ super-Yang-Mills theory [Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban'07]

The kinematics: $p_i^2 = 0$, $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_2 + p_3)^2 = -s - t$.

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F,G,H,I in progress.

$$F^D_{a_1,\ldots,a_{15}}(s,t;D) = \frac{1}{(i\pi^{D/2})^3} \int\int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{(-k_1^2)^{a_1}[-(p_2-k_1+k_2)^2]^{a_2}[-k_2^2]^{a_3}}$$
$$\times \frac{[-(k_1-k_3)^2]^{-a_{11}}[-(p_1+k_3)^2]^{-a_{12}}[-(p_1+k_2)^2]^{-a_{13}}}{[-(p_1+p_2+k_2)^2]^{a_4}[-k_3^2]^{a_5}[-(p_1+p_2+p_3+k_2-k_3)^2]^{a_6}}$$
$$\times \frac{[-(p_3+k_1)^2]^{-a_{14}}[-(p_3+k_2)^2]^{-a_{15}}}{(-(p_1+k_1)^2)^{a_7}(-(k_1-k_2)^2)^{a_8}[-(k_2-k_3)^2]^{a_9}[-(k_3-p_3)^2]^{a_{10}}}\,.$$

Partial results:
master integrals for D apart from the top sector [R.N. Lee'14]

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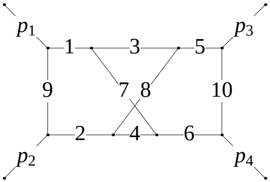
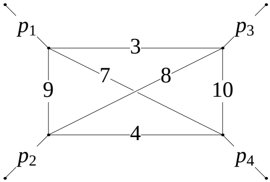
K_4 as a part of C [J. Henn, A.&V. Smirnov'13]





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Results expressed in terms of HPL
$H_{a_1,a_2,\ldots,a_n}(x)$, $a_i = 1, 0, -1$,
[E. Remiddi and J.A.M. Vermaseren'00]

B,C,D

IBP reduction by `FIRE` and by a private code by Bernhard Mistlberger.

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In all the cases, initial DE are transformed into

$$\partial_x f(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x}\right] f(x,\epsilon)\,.$$

where a and b are constant matrices.

Boundary conditions.

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Three singular points, at $x = 0$, $x = -1$, and $x = \infty$, corresponding to the limits $s \to 0$, $u \to 0$, and $t \to 0$, respectively.

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There is no this condition in the non-planar cases because non-planar diagrams have singularities in all the three channels.

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Typical contributions to the asymptotic expansion in the limit $x = t/x \to 0$:
hard-hard-hard contribution,
collinear-collinear-collinear contribution,
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The code `asy.m`
[A. Pak and A. Smirnov'10, B. Jantzen, A.S. and V.S.'12]
(which is now included into `FIESTA` [A.S.'09-15])
$\to$ expression of contributions of regions
[M. Beneke & V. S.'12] in terms of parametric integrals.

Three last elements of the basis

$$-\epsilon^6 s(s+t)(2sF_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0} - sF_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1}$$
$$-F_{1,1,0,0,1,1,1,1,1,1,0,0,0,0,0} + F_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1})\ ,$$
$$\epsilon^6 st(3F_{1,1,0,0,1,1,1,1,1,1,0,0,0,0,0} - 2F_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-1}$$
$$-F_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1})\ ,$$
$$\epsilon^6 s(-\frac{3}{2}s^2F_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0} + \frac{3}{2}s^2F_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1}$$
$$-\frac{9}{4}sF_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,-1} + \frac{5}{4}sF_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-1}$$
$$-2sF_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1} + \frac{3}{2}sF_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-2}$$
$$-5F_{1,1,1,-1,1,1,1,1,1,1,0,0,0,0,-1} + 4F_{1,1,1,0,1,1,1,1,1,1,0,0,-1,0,-1}$$
$$+3F_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-2} - 2F_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-2})\ .$$

Our analytical result for element 28 is

```
-(1/3) - (I ep \[Pi])/2 + (10 ep^2 \[Pi]^2)/9 +
 23/24 I ep^3 \[Pi]^3 - (271 ep^4 \[Pi]^4)/4320 - (
 10201 I ep^5 \[Pi]^5)/2880 - (23819 ep^6 \[Pi]^6)/20160 +
 1/2 ep H[{-1}, x] - 7/24 ep^3 \[Pi]^2 H[{-1}, x] -
 35/12 I ep^4 \[Pi]^3 H[{-1}, x] - 3809/960 ep^5 \[Pi]^4 H[{-1}, x] -
 1157/72 I ep^6 \[Pi]^5 H[{-1}, x] + 1/2 ep H[{0}, x] +
 1/2 I ep^2 \[Pi] H[{0}, x] - 61/24 ep^3 \[Pi]^2 H[{0}, x] +
 27/8 I ep^4 \[Pi]^3 H[{0}, x] - 103/576 ep^5 \[Pi]^4 H[{0}, x] + (
 58537 I ep^6 \[Pi]^5 H[{0}, x])/2880 +
 9/2 I ep^3 \[Pi] H[{-1, -1}, x] -
 35/12 ep^4 \[Pi]^2 H[{-1, -1}, x] -
 683/24 I ep^5 \[Pi]^3 H[{-1, -1}, x] +
 3361/240 ep^6 \[Pi]^4 H[{-1, -1}, x] - 1/2 ep^2 H[{-1, 0}, x] -
 5/2 I ep^3 \[Pi] H[{-1, 0}, x] + 77/24 ep^4 \[Pi]^2 H[{-1, 0}, x] +
 395/24 I ep^5 \[Pi]^3 H[{-1, 0}, x] + (
 739 ep^6 \[Pi]^4 H[{-1, 0}, x])/2880 - 1/2 ep^2 H[{0, -1}, x] -
 97/24 ep^4 \[Pi]^2 H[{0, -1}, x] +
 77/4 I ep^5 \[Pi]^3 H[{0, -1}, x] + (1/2880)
 18691 ep^6 \[Pi]^4 H[{0, -1}, x] - 5/2 I ep^3 \[Pi] H[{0, 0}, x] +
 79/12 ep^4 \[Pi]^2 H[{0, 0}, x] -
 445/24 I ep^5 \[Pi]^3 H[{0, 0}, x] +
 73/240 ep^6 \[Pi]^4 H[{0, 0}, x] - 9/2 ep^3 H[{-1, -1, -1}, x] +...
```

Evaluating planar three-loop vertex integrals at threshold.
[J. Henn, A. Smirnov and V. Smirnov'15]

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Numerical evaluation of planar and non-planar three-loop threshold integrals with FIESTA [P. Marquard, J.H. Piclum, D. Seidel and M. Steinhauser'14]
(evaluating NRQCD/QCD matching coefficients)

$$F_{a_1,\ldots,a_{12}} = \int\int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{[m^2-(k1+p1)^2]^{a_1}[m^2-(k_2+p_1)^2]^{a_2}}$$
$$\times \frac{1}{[m^2-(k_3+p_1)^2]^{a_3}[m^2-(k_3+p_2)^2]^{a_4}[m^2-(k_2+p_2)^2]^{a_5}}$$
$$\times \frac{1}{[m^2-(k_1+p_2)^2]^{a_6}[-k_1^2]^{a_7}[-(k_1-k_2)^2]^{a_8}[-(k_2-k_3)^2]^{a_9}}$$
$$\times \frac{1}{[-(k_1-k_3)^2]^{a_{10}}[-k_2^2]^{-a_{11}}[-k_3^2]^{-a_{12}}}$$

at $p_1^2 = m^2, p_2^2 = m^2, q^2 = (p_1 - p_2)^2 = 4m^2.$

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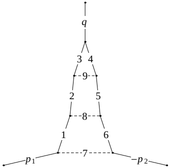
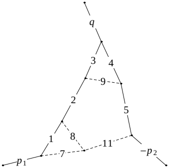
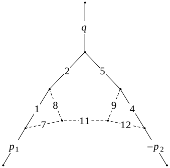
at $p_1^2 = m^2, p_2^2 = m^2, q^2 = (p_1 - p_2)^2 = 4m^2$.
Each index can be positive but the total number of positive indices cannot be more than 9. This family of integrals can be represented as the union of eight subfamilies.


q
3
4
9
2
5
8
1
6
7
p_1
$-p_2$

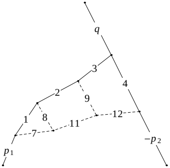
(1)


q
3
4
9
2
5
1
8
11
7
p_1
$-p_2$

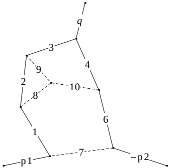
(2)


q
2
5
1
8
9
4
7
11
12
p_1
$-p_2$

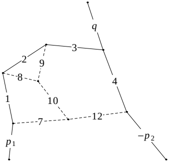
(3)


q
3
4
2
9
1
8
12
7
11
p_1
$-p_2$

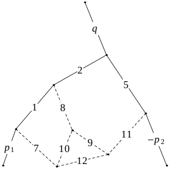
(4)


q
3
9
4
2
10
8
6
1
7
p1
−p2

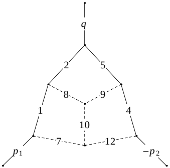
(5)


q
3
2
9
8
4
1
10
7
12
p_1
$-p_2$

(6)


q
2
5
1
8
11
9
7
10
12
p_1
$-p_2$

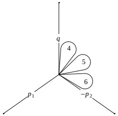
(7)

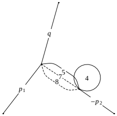

q
2
5
8
9
1
4
10
7
12
p_1
$-p_2$

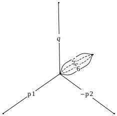
(8)

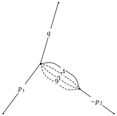
51 master integrals

$$
\begin{aligned}
&F_{0,0,0,1,1,1,0,0,0,0,0,0},\, F_{0,0,0,0,1,1,0,0,0,1,0,1},\, F_{0,0,0,0,1,1,0,0,1,1,0,0},\, F_{0,0,1,0,0,0,0,1,0,1,1,0},\\
&F_{0,0,1,0,0,1,0,1,1,0,0,0},\, F_{0,0,1,0,0,1,0,1,2,0,0,0},\, F_{0,0,1,0,1,1,0,0,0,1,0,0},\, F_{0,0,0,0,0,1,0,1,0,1,1,1},\\
&F_{0,0,1,0,0,1,0,1,0,1,1,0},\, F_{0,0,1,0,0,1,0,1,0,1,2,0},\, F_{0,0,1,0,0,1,0,1,0,2,1,0},\, F_{0,0,1,0,0,1,0,1,1,0,0,1},\\
&F_{0,0,1,0,0,1,0,1,1,0,1,0},\, F_{0,0,1,0,0,1,0,1,1,0,2,0},\, F_{0,0,1,0,0,1,0,2,1,0,1,0},\, F_{0,0,1,0,1,1,0,0,0,1,0,1},\\
&F_{0,0,1,0,1,1,0,0,1,1,0,0},\, F_{0,0,1,0,1,1,0,0,1,2,0,0},\, F_{0,1,1,0,0,0,1,1,0,1,0,0},\, F_{0,0,1,0,0,1,0,1,0,1,1,1},\\
&F_{0,0,1,0,0,1,0,1,1,1,1,0},\, F_{0,0,1,0,1,1,0,1,0,1,1,0},\, F_{0,0,1,0,1,1,0,1,0,1,2,0},\, F_{0,0,1,0,1,2,0,1,0,1,1,0},\\
&F_{0,0,1,0,1,2,0,0,1,1,1,0},\, F_{0,0,1,1,0,1,0,1,1,0,2,0},\, F_{0,0,1,1,0,1,1,1,1,0,0,0},\, F_{0,1,1,0,0,1,0,1,0,1,0,1},\\
&F_{0,1,1,0,0,1,0,1,0,2,0,1},\, F_{0,1,1,0,0,1,0,2,0,1,0,1},\, F_{0,1,1,0,0,2,0,1,0,1,0,1},\, F_{0,1,1,0,0,1,1,1,0,1,0,0},\\
&F_{0,1,1,0,1,0,1,0,1,1,0,0},\, F_{0,1,1,0,1,0,1,1,0,1,0,0},\, F_{0,1,1,0,1,1,0,0,1,0,0,1},\, F_{0,1,1,0,1,1,0,0,1,1,0,0},\\
&F_{0,0,1,0,1,1,0,1,0,1,1,1},\, F_{0,0,2,0,1,1,0,1,0,1,1,1},\, F_{0,0,1,1,1,1,0,1,0,1,1,0},\, F_{0,0,1,1,1,1,0,1,0,1,2,0},\\
&F_{0,1,1,0,1,0,1,1,1,0,0,1},\, F_{0,1,1,0,1,1,0,1,0,1,0,1},\, F_{0,1,1,0,1,1,0,1,0,1,0,2},\, F_{0,1,1,0,1,1,0,1,0,2,0,1},\\
&F_{0,1,1,0,1,1,0,2,0,1,0,1},\, F_{0,1,1,0,1,1,1,1,0,1,0,0},\, F_{0,1,1,0,1,1,1,1,0,2,0,0},\, F_{0,1,1,1,1,1,0,1,0,1,0,0},\\
&F_{1,1,1,0,0,0,0,1,0,1,1,1},\, F_{0,1,1,0,1,1,1,1,1,0,0,1},\, F_{0,1,1,0,1,1,1,1,1,1,0,1}\,.
\end{aligned}
$$


(1)


(2)


(3)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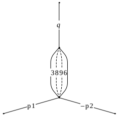
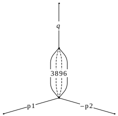
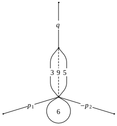
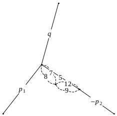

(5)


(6a)


(7)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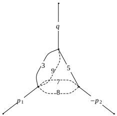
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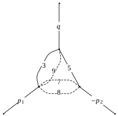
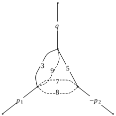
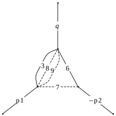

(10a)


(11b)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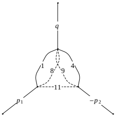
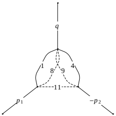
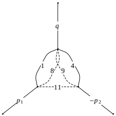
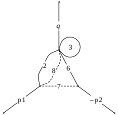

(13)


(14a)


(15b)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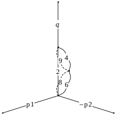
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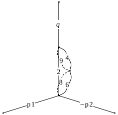
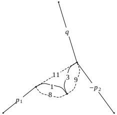
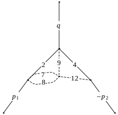

(18a)


(19)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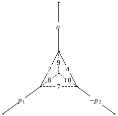
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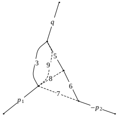
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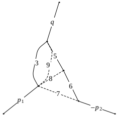
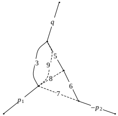

(23a)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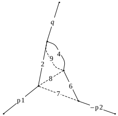
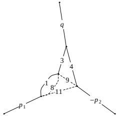
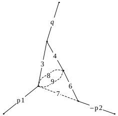
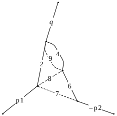
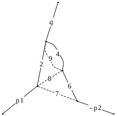
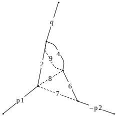

(25)

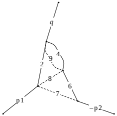

(26)

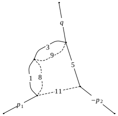

(27)


(28)


(29a)


(30b)


(31c)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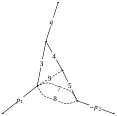
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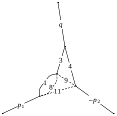
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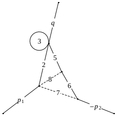
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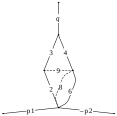
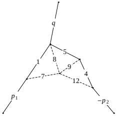
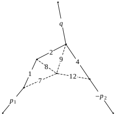
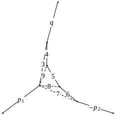

(36)


(37)


(38a)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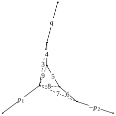
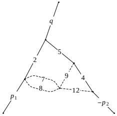
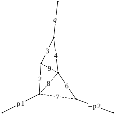
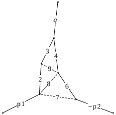

(40a)

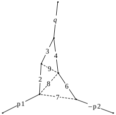

(41)



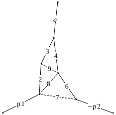
(42)



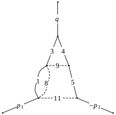
(43a)



(44b)



(45c)



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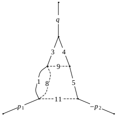

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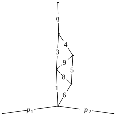

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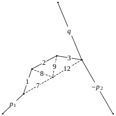
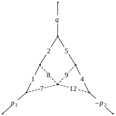


(49)



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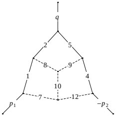

(51)

Our goal are integrals at $s = q^2 \equiv (p_2 - p_2)^2 = 4m^2$.

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Turn to the corresponding family of integrals at general q^2 and introduce

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}$$

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The values $x = 0$ and $x = -1$ correspond to $s = 0$ and $s = 4m^2$.

DE

$$f'(\epsilon, x) = \epsilon \tilde{A}'(x) f(x, \epsilon),$$

where $\tilde{A} = \sum_k \tilde{A}_{\alpha_k} \log(\alpha_k)$ and the letters α_k are $x, 1+x, 1-x, 1+x+x^2$.

90 elements of this basis $f(x)$ are

$$\Big\{F_{0,0,0,3,3,3,0,0,0,0,0,0}\,, \quad \varepsilon\frac{x^2-1}{x}F_{0,0,2,1,3,3,0,0,0,0,0,0}, \ldots$$
$$\varepsilon^6\frac{(1-x^2)^2}{x^2}F_{1,0,1,1,1,1,1,1,1,0,0,0}\,, \quad (1-2\varepsilon)\varepsilon^4 F_{1,2,1,0,0,0,1,1,1,0,0,1}\Big\}$$

90 elements of this basis $f(x)$ are

$$\{F_{0,0,0,3,3,3,0,0,0,0,0,0}\ ,\quad \varepsilon\frac{x^2-1}{x}F_{0,0,2,1,3,3,0,0,0,0,0,0},\dots$$
$$\varepsilon^6\frac{(1-x^2)^2}{x^2}F_{1,0,1,1,1,1,1,1,1,0,0,0}\ ,\quad (1-2\varepsilon)\varepsilon^4F_{1,2,1,0,0,0,1,1,1,0,0,1}\}$$

A solution in an epsilon-expansion with coefficients written in terms of Goncharov (multiple) polylogarithms (GPL)

$$G(a_1,\dots,a_n;z)=\int_0^z\frac{\mathrm{d}t}{t-a_1}G(a_2,\dots,a_n;t)$$

with indices a_i taken from the seven-letters alphabet $\{0,r_1,r_3,-1,r_4,r_2,1\}$ with

$$r_{1,2}=\frac{1}{2}\left(1\pm\sqrt{3}\,\mathrm{i}\right)\ ,\quad r_{3,4}=\frac{1}{2}\left(-1\pm\sqrt{3}\,\mathrm{i}\right)\ .$$

A typical expression for analytical results for the elements of the basis

```
ep^4*(-24*G[{-1}, 1]*G[{0}, x]*G[{0, -1}, 1] +
   24*G[{0, -1}, 1]*G[{0, -1}, x] - 23*G[{0, -1}, 1]*G[{0, 0}, x] -
   12*G[{-1}, 1]*G[{0}, x]*G[{0, 1}, 1] + 12*G[{0, -1}, x]*G[{0, 1}, 1] -
   (23*G[{0, 0}, x]*G[{0, 1}, 1])/2 + 12*G[{0, -1}, 1]*G[{0, 1}, x] +
   6*G[{0, 1}, 1]*G[{0, 1}, x] + 12*G[{0, -1}, 1]*G[{1, 0}, x] +
   6*G[{0, 1}, 1]*G[{1, 0}, x] - 9*G[{0, -1}, 1]*G[{r1, 0}, x] -
   (9*G[{0, 1}, 1]*G[{r1, 0}, x])/2 - 9*G[{0, -1}, 1]*G[{r2, 0}, x] -
   (9*G[{0, 1}, 1]*G[{r2, 0}, x])/2 + 24*G[{0}, x]*G[{-1, 0, -1}, 1] +
   12*G[{0}, x]*G[{-1, 0, 1}, 1] + 24*G[{0}, x]*G[{0, -1, -1}, 1] +
   24*G[{-1}, x]*G[{0, 0, -1}, 1] - 48*G[{0}, x]*G[{0, 0, -1}, 1] +
   48*G[{1}, x]*G[{0, 0, -1}, 1] - 18*G[{r1}, x]*G[{0, 0, -1}, 1] -
   18*G[{r2}, x]*G[{0, 0, -1}, 1] + 24*G[{-1}, x]*G[{0, 0, 1}, 1] -
   (57*G[{0}, x]*G[{0, 0, 1}, 1])/2 + 24*G[{1}, x]*G[{0, 0, 1}, 1] -
   (21*G[{r1}, x]*G[{0, 0, 1}, 1])/2 - (21*G[{r2}, x]*G[{0, 0, 1}, 1])/2 -
   6*G[{0}, x]*G[{0, 1, 1}, 1] - 24*G[{-1, -1, 0, 0}, x] +
   36*G[{-1, 0, 0, 0}, x] - 24*G[{-1, 1, 0, 0}, x] +
   24*G[{0, -1, -1, 0}, x] + 2*G[{0, -1, 0, 0}, x] + 12*G[{0, -1, 1, 0}, x] -
   23*G[{0, 0, -1, 0}, x] - (23*G[{0, 0, 1, 0}, x])/2 +
   12*G[{0, 1, -1, 0}, x] + (11*G[{0, 1, 0, 0}, x])/2 +
   6*G[{0, 1, 1, 0}, x] - 24*G[{1, -1, 0, 0}, x] + 12*G[{1, 0, -1, 0}, x] +
   15*G[{1, 0, 0, 0}, x] + 6*G[{1, 0, 1, 0}, x] - 12*G[{1, 1, 0, 0}, x] -
   9*G[{r1, 0, -1, 0}, x] + 6*G[{r1, 0, 0, 0}, x] -
   (9*G[{r1, 0, 1, 0}, x])/2 + (3*G[{r1, 1, 0, 0}, x])/2 -
   9*G[{r2, 0, -1, 0}, x] + 6*G[{r2, 0, 0, 0}, x] -
   (9*G[{r2, 0, 1, 0}, x])/2 + (3*G[{r2, 1, 0, 0}, x])/2 +
   (3*G[{0}, x]*Zeta[3])/2 - (3*G[{r1}, x]*Zeta[3])/2 -
   (3*G[{r2}, x]*Zeta[3])/2 -
   (3*(16*G[{0, -1}, 1]^2 + 8*G[{0, -1}, 1]*G[{0, 1}, 1] +...
```

Threshold expansion

$$F(a_1,\ldots,a_{12};q^2,m^2) \sim \sum_{n=n_0}^{\infty}\sum_{j=0}^{3}(4m^2-q^2)^{n-j\epsilon}F_{n,j}(a_1,\ldots,a_{12};q^2)\,.$$

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Our goal are one-scale integrals $F_{0,0}(a_1, \ldots, a_{12}; m^2)$ defined with q^2 set to $4m^2$.

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Expand 'naively' in $x+1$ the corresponding integrals. Introduce one more (13th) index for the order of this derivative in s, i.e. deal with the family

$$F'(a_1,\ldots,a_{12},a_{13}) = \left(\frac{\partial}{\partial s}\right)^{-a_{13}} F(a_1,\ldots,a_{12})\Big|_{s=4m^2}$$

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They are all with $a_{13} = 0$, i.e directly correspond to the 51 master threshold integrals.

Matching at threshold

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$x = y - 1,\ y \to 0$:

$$f'(\epsilon, y) = \epsilon \frac{\tilde{A}'(y)}{y} f(\epsilon, y)\,,$$

where $\tilde{A}'(y) = A_0 + yA_1 + y^2 A_2 + \ldots$.

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In the language of differential equations, the naive part of the expansion near $y = 0$ corresponds to zero eigenvalues of the matrix A_0 while eigenvalues proportional to ε correspond to other contributions.

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Construct a polynomial $P = 1 + \sum_{r=1} P_r y^r$ such that the DE for the function g defined by $f = Pg$ takes the form $yg'(y) = A_0 g(y)$ (with A_0 is independent of y).

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Then the solution of this equation is just $g = y^{A_0} g_0$ with a boundary value g_0.

We implemented this algorithm and constructed P_r up to $r = 5$.

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Solving these equations $\to$ coefficients of the epsilon expansion of the MI up to some order written in terms of GPL $G(a_1, \ldots, a_n; 1)$ with $a_1 \neq 1$ and a_i taken from the alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$.

Equating the part of our analytic results for the basis without $\log(x+1)$ and the naive part of the threshold expansion expressed in terms of the 51 threshold MI.

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Examples of our results
[J. Henn, A. Smirnov and V. Smirnov'15]

$$F_{0,0,1,0,1,1,0,1,0,1,1,1} = -\frac{27}{2}\log(2)G_R(0,0,r_2,-1) - \frac{181\zeta(5)}{32} - \frac{21}{2}\log^2(2)\zeta(3)$$
$$+\frac{115\pi^2\zeta(3)}{48} - 12\text{Li}_5\left(\frac{1}{2}\right) - 12\log(2)\text{Li}_4\left(\frac{1}{2}\right) - \frac{2\log^5(2)}{5} + \frac{1}{6}\pi^2\log^3(2)$$
$$-\frac{81}{8}G_R(0,0,r_4,1)\log(2) + \frac{277}{960}\pi^4\log(2)\ ,$$
$$F_{0,0,1,1,1,1,0,1,0,1,1,0} = -\frac{27}{4}\log(2)G_R(0,0,r_2,-1) - \frac{341\zeta(5)}{64} - \frac{21}{4}\log^2(2)\zeta(3)$$
$$+\frac{211\pi^2\zeta(3)}{96} - 6\text{Li}_5\left(\frac{1}{2}\right) - 6\log(2)\text{Li}_4\left(\frac{1}{2}\right) - \frac{\log^5(2)}{5}$$
$$+\frac{1}{12}\pi^2\log^3(2) - \frac{81}{16}G_R(0,0,r_4,1)\log(2) + \frac{277\pi^4\log(2)}{1920}\ ,$$
$$F_{0,0,1,1,1,1,0,1,0,1,2,0} = -\frac{1}{24\epsilon^3} + \frac{1}{3\epsilon^2} - \frac{25\pi^2}{96\epsilon} - \frac{13}{6\epsilon} - \frac{97\zeta(3)}{24}$$
$$-\pi^2\log(2) + \frac{7\pi^2}{4} + \frac{40}{3}\ ,\ldots$$

$$G(a_1, \ldots, a_n; 1) = G_R(a_1, \ldots, a_n) + \mathrm{i}\, G_I(a_1, \ldots, a_n)$$

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Constants present in results for Feynman integrals up to weight 5 were discussed in [Fleischer and M. Kalmykov'99, Davydychev M. Kalmykov'00, M. Kalmykov and B. Kniehl'10].

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For example,

$$
\begin{aligned}
G_I(r_2) &= -\frac{\pi}{3}\,, \quad G_R(-1) = \log(2)\,, \\
G_R(0,0,1) &= -\zeta(3)\,, \quad G_R(0,0,0,1) = -\frac{\pi^4}{90}\,, \\
G_R(0,0,0,0,1) &= -\zeta(5)\,, \\
G_R(0,0,1,1,-1) &= -2\mathrm{Li}_5\left(\frac{1}{2}\right) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{\pi^2\zeta(3)}{96} \\
&\quad + \frac{151\zeta(5)}{64} - \frac{\log^5(2)}{15} + \frac{1}{18}\pi^2\log^3(2) - \frac{1}{96}\pi^4\log(2)\,.
\end{aligned}
$$

Shuffle relations

$$G(a_1, \ldots, a_{n_1}; x)\, G(b_1, \ldots, b_{n_2}; x) = \sum_{c = a \uplus b} G(c_1, \ldots, c_{n_1+n_2}; x)\ ,$$

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Zhao's conjecture [J. Zhao'07]: *all independent (polynomial) relations among GPL at nth roots of unity are shuffle, stuffle, regularization, distributions relations and seeded relations, and lifted relations thereof.*

In our case, $n = 6$, we used the first four of the above type of relations and the complex conjugation relations

$$G(a_1^*, \ldots, a_n^*; 1) = G(a_1, \ldots, a_n; 1)^*$$

with $r_1^* = r_2, r_3^* = r_4$.

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with $r_1^* = r_2, r_3^* = r_4$.

The total number of these five sets of relations grows fast when the weight is increased. At weight 6, we have 654452 equations for the real parts and 654937 equations for the imaginary parts of $G(a_1, \ldots, a_n; 1)$.

We solved these relations up to weight 5 and hope to do this for weight 6.

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It turns out that the resulting constants, independent in the sense of these relations are still linear dependent, i.e. one can find additional relations for genuine constants of a given weight $G(a_1, \ldots, a_n; 1)$, i.e. linearly express them in terms of a smaller set of such constants and products of constants of lower weights.

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We did this with experimental mathematics using the `PSLQ` algorithm [H.R.P. Ferguson, D.H. Bailey, and S. Arno] and `ginac` [C. Bauer, A. Frink and R. Kreckel] to evaluate GPLs with a big accuracy.

Our basis for the real parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

```
{GR[0, 0, r2, -1], GR[0, 0, r4, 1], GR[r2, 1, 1, -1],
 GR[r2, 1, 1, r3], GR[r2, 1, r2, -1]}
```

and 25 products of constants of lower weights

```
{GR[-1]^4, GI[r2]^2 GR[-1]^2, GI[r2]^4, GR[-1]^3 GR[r4],
 GI[r2]^2 GR[-1] GR[r4], GR[-1]^2 GR[r4]^2, GI[r2]^2 GR[r4]^2,
 GR[-1] GR[r4]^3, GR[r4]^4, GI[r2] GI[0, r2] GR[-1],
 GI[r2] GI[0, r2] GR[r4], GI[0, r2]^2, GR[-1]^2 GR[r2, -1],
 GI[r2]^2 GR[r2, -1], GR[-1] GR[r4] GR[r2, -1], GR[r4]^2 GR[r2, -1],
 GR[r2, -1]^2, GR[-1] GR[0, 0, 1], GR[r4] GR[0, 0, 1],
 GI[r2] GI[0, 1, r4], GI[r2] GI[0, r2, -1], GR[-1] GR[r2, 1, -1],
 GR[r4] GR[r2, 1, -1], GR[-1] GR[r2, 1, r3], GR[r4] GR[r2, 1, r3]}
```

Our basis for the imaginary parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

```
{GI[0, 0, 0, r2], GI[0, 1, 1, r4], GI[0, 1, r2, -1], GI[0, 1, r2, r3],
  GI[0, r2, 1, -1]}
```

and 20 products of constants of lower weights

```
{GI[r2] GR[-1]^3, GI[r2]^3 GR[-1], GI[r2] GR[-1]^2 GR[r4],
 GI[r2]^3 GR[r4], GI[r2] GR[-1] GR[r4]^2, GI[r2] GR[r4]^3,
 GI[0, r2] GR[-1]^2, GI[r2]^2 GI[0, r2], GI[0, r2] GR[-1] GR[r4],
 GI[0, r2] GR[r4]^2, GI[r2] GR[-1] GR[r2, -1],
 GI[r2] GR[r4] GR[r2, -1], GI[0, r2] GR[r2, -1], GI[r2] GR[0, 0, 1],
 GI[0, 1, r4] GR[-1], GI[0, 1, r4] GR[r4], GI[0, r2, -1] GR[-1],
 GI[0, r2, -1] GR[r4], GI[r2] GR[r2, 1, -1], GI[r2] GR[r2, 1, r3]}
```

Our basis for the real parts of $G(a_1, \ldots, a_5; 1)$ consists of 13 constants of weight 5

```
{GR[0, 0, 0, 0, 1], GR[0, 0, 1, 1, -1], GR[0, 0, 1, 1, r4],
 GR[0, 0, 1, r2, -1], GR[0, 0, 1, r2, r3], GR[0, 0, 1, r2, r4],
 GR[0, 0, r2, 1, -1], GR[r2, 1, 1, -1, r2], GR[r2, 1, 1, 1, -1],
 GR[r2, 1, 1, 1, r3], GR[r2, 1, 1, r2, -1], GR[r2, 1, 1, r2, r3],
 GR[r2, 1, 1, r4, -1]}
```

and 63 products of constants of lower weights

```
{GR[-1]^5, GI[r2]^2 GR[-1]^3, GI[r2]^4 GR[-1], GR[-1]^4 GR[r4],
 GI[r2]^2 GR[-1]^2 GR[r4], GI[r2]^4 GR[r4], GR[-1]^3 GR[r4]^2,
 GI[r2]^2 GR[-1] GR[r4]^2, GR[-1]^2 GR[r4]^3, GI[r2]^2 GR[r4]^3,
 GR[-1] GR[r4]^4, GR[r4]^5, GI[r2] GI[0, r2] GR[-1]^2,
 GI[r2]^3 GI[0, r2], GI[r2] GI[0, r2] GR[-1] GR[r4],
 GI[r2] GI[0, r2] GR[r4]^2, GI[0, r2]^2 GR[-1], GI[0, r2]^2 GR[r4],
 GR[-1]^3 GR[r2, -1], GI[r2]^2 GR[-1] GR[r2, -1],
 GR[-1]^2 GR[r4] GR[r2, -1], GI[r2]^2 GR[r4] GR[r2, -1],
 GR[-1] GR[r4]^2 GR[r2, -1], GR[r4]^3 GR[r2, -1],
 GI[r2] GI[0, r2] GR[r2, -1], GR[-1] GR[r2, -1]^2,
 GR[r4] GR[r2, -1]^2, GR[-1]^2 GR[0, 0, 1], GI[r2]^2 GR[0, 0, 1],
 GR[-1] GR[r4] GR[0, 0, 1], GR[r4]^2 GR[0, 0, 1],
 GR[r2, -1] GR[0, 0, 1], GI[r2] GI[0, 1, r4] GR[-1],
 GI[r2] GI[0, 1, r4] GR[r4], GI[0, r2] GI[0, 1, r4],
 GI[r2] GI[0, r2, -1] GR[-1], GI[r2] GI[0, r2, -1] GR[r4],
 GI[0, r2] GI[0, r2, -1], GR[-1]^2 GR[r2, 1, -1],
 GI[r2]^2 GR[r2, 1, -1], GR[-1] GR[r4] GR[r2, 1, -1],
 GR[r4]^2 GR[r2, 1, -1], GR[r2, -1] GR[r2, 1, -1],
 GR[-1]^2 GR[r2, 1, r3], GI[r2]^2 GR[r2, 1, r3],
 GR[-1] GR[r4] GR[r2, 1, r3], GR[r4]^2 GR[r2, 1, r3],
 GR[r2, -1] GR[r2, 1, r3], GI[r2] GI[0, 0, 0, r2],
 GR[-1] GR[0, 0, r2, -1], GR[r4] GR[0, 0, r2, -1],
 GR[-1] GR[0, 0, r4, 1], GR[r4] GR[0, 0, r4, 1],
 GI[r2] GI[0, 1, 1, r4], GI[r2] GI[0, 1, r2, -1],
 GI[r2] GI[0, 1, r2, r3], GI[r2] GI[0, r2, 1, -1],
 GR[-1] GR[r2, 1, 1, -1], GR[r4] GR[r2, 1, 1, -1],
 GR[-1] GR[r2, 1, 1, r3], GR[r4] GR[r2, 1, 1, r3],
 GR[-1] GR[r2, 1, r2, -1], GR[r4] GR[r2, 1, r2, -1]}
```

Our basis for the imaginary parts of $G(a_1, \ldots, a_5; 1)$ consists of 11 constants of weight 5

```
{GI[0, 0, 0, 1, r2], GI[0, 0, 0, 1, r4], GI[0, 0, 0, r2, -1],
GI[0, 1, 1, -1, r2], GI[0, 1, 1, -1, r4], GI[0, 1, 1, 1, r4],
GI[0, 1, 1, r2, r3], GI[0, 1, 1, r4, -1], GI[0, 1, 1, r4, r1],
GI[0, 1, r2, r3, r2], GI[0, r2, 1, 1, -1]}
```

and 57 products of constants of lower weights

```
{GI[r2] GR[-1]^4, GI[r2]^3 GR[-1]^2, GI[r2]^5, GI[r2] GR[-1]^3 GR[r4],
 GI[r2]^3 GR[-1] GR[r4], GI[r2] GR[-1]^2 GR[r4]^2, GI[r2]^3 GR[r4]^2,
 GI[r2] GR[-1] GR[r4]^3, GI[r2] GR[r4]^4, GI[0, r2] GR[-1]^3,
GI[r2]^2 GI[0, r2] GR[-1], GI[0, r2] GR[-1]^2 GR[r4],
GI[r2]^2 GI[0, r2] GR[r4], GI[0, r2] GR[-1] GR[r4]^2,
GI[0, r2] GR[r4]^3, GI[r2] GI[0, r2]^2, GI[r2] GR[-1]^2 GR[r2, -1],
GI[r2]^3 GR[r2, -1], GI[r2] GR[-1] GR[r4] GR[r2, -1],
GI[r2] GR[r4]^2 GR[r2, -1], GI[0, r2] GR[-1] GR[r2, -1],
GI[0, r2] GR[r4] GR[r2, -1], GI[r2] GR[r2, -1]^2,
GI[r2] GR[-1] GR[0, 0, 1], GI[r2] GR[r4] GR[0, 0, 1],
GI[0, r2] GR[0, 0, 1], GI[0, 1, r4] GR[-1]^2, GI[r2]^2 GI[0, 1, r4],
GI[0, 1, r4] GR[-1] GR[r4], GI[0, 1, r4] GR[r4]^2,
GI[0, 1, r4] GR[r2, -1], GI[0, r2, -1] GR[-1]^2,
GI[r2]^2 GI[0, r2, -1], GI[0, r2, -1] GR[-1] GR[r4],
GI[0, r2, -1] GR[r4]^2, GI[0, r2, -1] GR[r2, -1],
GI[r2] GR[-1] GR[r2, 1, -1], GI[r2] GR[r4] GR[r2, 1, -1],
GI[0, r2] GR[r2, 1, -1], GI[r2] GR[-1] GR[r2, 1, r3],
GI[r2] GR[r4] GR[r2, 1, r3], GI[0, r2] GR[r2, 1, r3],
GI[0, 0, 0, r2] GR[-1], GI[0, 0, 0, r2] GR[r4],
GI[r2] GR[0, 0, r2, -1], GI[r2] GR[0, 0, r4, 1],
GI[0, 1, 1, r4] GR[-1], GI[0, 1, 1, r4] GR[r4],
GI[0, 1, r2, -1] GR[-1], GI[0, 1, r2, -1] GR[r4],
GI[0, 1, r2, r3] GR[-1], GI[0, 1, r2, r3] GR[r4],
GI[0, r2, 1, -1] GR[-1], GI[0, r2, 1, -1] GR[r4],
GI[r2] GR[r2, 1, 1, -1], GI[r2] GR[r2, 1, 1, r3],
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to be continued

BACKUP SLIDES

Lee [R.N. Lee'14]: transition to UT basis in three steps (in the case of one variable).

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- Normalizing eigenvalues of the matrices which are coefficients the Fuchsian singularities when one tries to make them proportional to ϵ.
- Providing a linear dependence on ϵ.

At each of the three steps, one is looking for a proper linear transformation of the current basis. The first two steps are based on the so-called balance transformation

$$\mathcal{B}(\mathbb{P}, x_1, x_2|x) = \overline{\mathbb{P}} + c\frac{x - x_2}{x - x_1}\mathbb{P}\,,$$

where c is a constant, $\mathbb{P}$, $\overline{\mathbb{P}}$ are the two complementary projectors, i.e. $\mathbb{P}^2 = \mathbb{P}$ and $\overline{\mathbb{P}} = \mathbb{I} - \mathbb{P}$.

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The idea of using a balance transformation is that with its help one can take care of one singular point x_1 (by providing a Fuchsian singularity or by normalizing eigenvalues corresponding to a given singular point) and not to spoil these properties at a second singular point, x_2.