Evaluating Feynman integrals by uniformly transcendental differential equations

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Based on collaboration with Johannes Henn, Bernhard Miltlberger and Alexander Smirnov
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- Introduction. The method of differential equations
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- Introduction. The method of differential equations
- Evaluating non-planar on-shell three-loop four-point massless integrals
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More on the method of differential equations in other talks
[L. Tancredi, O. Gituliar, B. Mistlberger, R. Schabinger, C. Papadopoulos, A. von Manteuffel, C. Wever]
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Introduction. The method of differential equations

[A.V. Kotikov’91, E. Remiddi’97, T. Gehrmann & E. Remiddi’00, J. Henn’13]
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Solve DE
Let $f = (f_1, \ldots, f_N)$ be primary master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.
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DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x)f(\epsilon, x),$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each $A_i$ is an $N \times N$ matrix.
Henn (2013): turn to a new basis where DE take the form

\[ \partial_i f(\epsilon, x) = \epsilon A_i(x) f(\epsilon, x). \]
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In the differential form,

$$d f(\epsilon, x) = \epsilon (d \tilde{A}(x)) f(x, \epsilon),$$

where

$$\tilde{A} = \sum_k \tilde{A}_{\alpha_k} \log(\alpha_k).$$
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and $\tilde{A}_{\alpha_k}$ are constant matrices. The arguments of the logarithms $\alpha_i$ (letters) are functions of $x$. Elements of such basis turn out to be uniformly transcendental (UT).
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Let us call it epsilon form.
The case of two scales, i.e. with one variable in the DE, i.e. $n = 1$. 
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One tries to achieve the following form of DE:

$$f'(\epsilon, x) = \epsilon \sum_k \frac{a_k}{x - x^{(k)}} f(\epsilon, x).$$

where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices $a_k$ are independent of $x$ and $\epsilon$. 
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where \( x^{(k)} \) is the set of singular points of the DE and \( N \times N \) matrices \( a_k \) are independent of \( x \) and \( \epsilon \).

For example, if \( x_k = 0, -1, 1 \) then results for elements of such a basis are expressed in terms of HPL.
How to turn to a UT basis?
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- In simple situations where integrals can be expressed in terms of gamma functions, just adjust indices properly
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- An approach using Magnus and Dyson series expansion
A part of the procedure is algorithmically described in [T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs’14]
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Constructing UT elements of the basis at the level of integrand [Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz and J. Trnka’14]
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In the case of one variables was algorithmically described [R.N. Lee’14]
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Evaluating non-planar on-shell three-loop four-point massless integrals

(a) \hspace{2cm} (b) \hspace{2cm} (c)

(d) \hspace{2cm} (e) \hspace{2cm} (f)

(g) \hspace{2cm} (h) \hspace{2cm} (i)
The kinematics: $p_i^2 = 0$, $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_2 + p_3)^2 = -s - t$. 
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B,C,D [J. Henn, B. Mistlberger and V. Smirnov’15]
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$$F^D_{a_1,\ldots,a_{15}}(s, t; D) = \frac{1}{(i\pi^{D/2})^3} \int \int \int \frac{d^D k_1 d^D k_2 d^D k_3}{(-k_1^2)^{a_1}(-(p_2 - k_1 + k_2)^2)^{a_2}(-k_2^2)^{a_3}} \times \frac{[-(k_1 - k_3)^2]^{-a_{11}}[-(p_1 + k_3)^2]^{-a_{12}}[-(p_1 + k_2)^2]^{-a_{13}}}{[-(p_1 + p_2 + k_2)^2]^{a_4}[-k_3^2]^{a_5}[-(p_1 + p_2 + p_3 + k_2 - k_3)^2]^{a_6}} \times \frac{[-(p_3 + k_1)^2]^{-a_{14}}[-(p_3 + k_2)^2]^{-a_{15}}}{(-(p_1 + k_1)^2)^{a_7}(-(k_1 - k_2)^2)^{a_8}[-(k_2 - k_3)^2]^{a_9}[-(k_3 - p_3)^2]^{a_{10}}}.$$
Partial results:
master integrals for $D$ apart from the top sector [R.N. Lee’14]
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Results expressed in terms of HPL
\(H_{a_1,a_2,\ldots,a_n}(x), a_i = 1, 0, -1\),
\([\text{E. Remiddi and J.A.M. Vermaseren'00}]\)
\( B, C, D \)

IBP reduction by FIRE and by a private code by Bernhard Mistlberger.
$B, C, D$

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In all the cases, initial DE are transformed into

$$\partial_x f(x, \epsilon) = \epsilon \left[ \frac{a}{x} + \frac{b}{1 + x} \right] f(x, \epsilon).$$

where $a$ and $b$ are constant matrices.
Boundary conditions.
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Three singular points, at $x = 0$, $x = -1$, and $x = \infty$, corresponding to the limits $s \to 0$, $u \to 0$, and $t \to 0$, respectively.
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There is no this condition in the non-planar cases because non-planar diagrams have singularities in all the three channels.
Studying limits, $s \to 0$, $t \to 0$, $u \to 0$. 
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Typical contributions to the asymptotic expansion in the limit \( x = t/x \to 0 \):
hard-hard-hard contribution,
collinear-collinear-collinear contribution,
ultrasoft-collinear-collinear contribution.
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The code \texttt{asy.m}
[A. Pak and A. Smirnov’10, B. Jantzen, A.S. and V.S.’12]
(which is now included into \textsc{FIESTA} [A.S.’09-15])
→ expression of contributions of regions
Three last elements of the basis

\[-\epsilon^6 s (s + t) (2sF_{1,1,0,1,1,1,1,1,1,0,0,0,0,0} - sF_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1})
\quad - F_{1,1,0,0,1,1,1,1,1,1,0,0,0,0,0,0} + F_{1,1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1})) ,
\]

\[\epsilon^6 st(3F_{1,1,0,0,1,1,1,1,1,1,0,0,0,0,0} - 2F_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0,-1})
\quad - F_{1,1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1})) ,
\]

\[\epsilon^6 s \left( -\frac{3}{2} s^2 F_{1,1,0,1,1,1,1,1,1,1,1,0,0,0,0,0} + \frac{3}{2} s^2 F_{1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1} \right)
\quad - \frac{9}{4} sF_{1,1,0,1,1,1,1,1,1,1,1,0,0,0,0,-1} + \frac{5}{4} sF_{1,1,1,0,1,1,1,1,1,1,1,0,0,0,0,-1} \right)
\quad - 2sF_{1,1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1}) + \frac{3}{2} sF_{1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,-2} \right)
\quad - 5F_{1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1} + 4F_{1,1,1,0,1,1,1,1,1,1,1,0,0,-1,0,-1} \right)
\quad + 3F_{1,1,1,0,1,1,1,1,1,1,1,0,0,0,0,-2} - 2F_{1,1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-2}) \right) .\]
Our analytical result for element 28 is

\[-(1/3) - (\text{I ep } \Pi/2 + (10 \text{ ep}^2 \Pi^2)/9 + \\
23/24 \text{ I ep}^3 \Pi^3 - (271 \text{ ep}^4 \Pi^4)/4320 - ( \\
10201 \text{ ep}^5 \Pi^5)/2880 - (23819 \text{ ep}^6 \Pi^6)/20160 + \\
1/2 \text{ ep } H\{-1\}, x\] - 7/24 \text{ ep}^3 \Pi^2 H\{-1\}, x\] - \\
35/12 \text{ I ep}^4 \Pi^3 H\{-1\}, x\] - 3809/960 \text{ ep}^5 \Pi^4 H\{-1\}, x\] - \\
1157/72 \text{ I ep}^6 \Pi^5 H\{-1\}, x\] + 1/2 \text{ ep } H\{0\}, x\] + \\
1/2 \text{ I ep}^2 \Pi H\{0\}, x\] - 61/24 \text{ ep}^3 \Pi^2 H\{0\}, x\] + \\
27/8 \text{ I ep}^4 \Pi^3 H\{0\}, x\] - 103/576 \text{ ep}^5 \Pi^4 H\{0\}, x\] + ( \\
58537 \text{ I ep}^6 \Pi^5 H\{0\}, x\)]/2880 + \\
9/2 \text{ I ep}^3 \Pi H\{-1, -1\}, x\] - \\
35/12 \text{ ep}^4 \Pi^2 H\{-1, -1\}, x\] - \\
683/24 \text{ I ep}^5 \Pi^3 H\{-1, -1\}, x\] + \\
3361/240 \text{ ep}^6 \Pi^4 H\{-1, -1\}, x\] - 1/2 \text{ ep}^2 H\{-1, 0\}, x\] - \\
5/2 \text{ I ep}^3 \Pi H\{-1, 0\}, x\] + 77/24 \text{ ep}^4 \Pi^2 H\{-1, 0\}, x\] + \\
395/24 \text{ I ep}^5 \Pi^3 H\{-1, 0\}, x\] + ( \\
739 \text{ ep}^6 \Pi^4 H\{-1, 0\}, x\)]/2880 - 1/2 \text{ ep}^2 H\{0, -1\}, x\] - \\
97/24 \text{ ep}^4 \Pi^2 H\{0, -1\}, x\] + \\
77/4 \text{ I ep}^5 \Pi^3 H\{0, -1\}, x\] + (1/2880) \\
18691 \text{ ep}^6 \Pi^4 H\{0, -1\}, x\] - 5/2 \text{ I ep}^3 \Pi H\{0, 0\}, x\] + \\
79/12 \text{ ep}^4 \Pi^2 H\{0, 0\}, x\] - \\
445/24 \text{ I ep}^5 \Pi^3 H\{0, 0\}, x\] + \\
73/240 \text{ ep}^6 \Pi^4 H\{0, 0\}, x\] - 9/2 \text{ ep}^3 H\{-1, -1, -1\}, x\] +...
Evaluating planar three-loop vertex integrals at threshold.

[J. Henn, A. Smirnov and V. Smirnov ’15]
Evaluating planar three-loop vertex integrals at threshold.
[J. Henn, A. Smirnov and V. Smirnov’15]
evaluating NRQCD/QCD matching coefficients)
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Evaluating planar three-loop vertex integrals at threshold

\[
F_{a_1, \ldots, a_{12}} = \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{[m^2 - (k_1 + p_1)^2]^{a_1} [m^2 - (k_2 + p_1)^2]^{a_2}} \times \frac{1}{[m^2 - (k_3 + p_1)^2]^{a_3} [m^2 - (k_3 + p_2)^2]^{a_4} [m^2 - (k_2 + p_2)^2]^{a_5}} \times \frac{1}{[m^2 - (k_1 + p_2)^2]^{a_6} [-(k_1^2)]^{a_7} [-(k_1 - k_2)^2]^{a_8} [-(k_2 - k_3)^2]^{a_9}} \times \frac{1}{[-(k_1 - k_3)^2]^{a_{10}} [-k_2^{a_{11}}] [-k_3^{a_{12}}]}
\]

at \( p_1^2 = m^2, p_2^2 = m^2, q^2 = (p_1 - p_2)^2 = 4m^2. \)
\[ F_{a_1, \ldots, a_{12}} = \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{[m^2 - (k_1 + p_1)^2]^{a_1} \, [m^2 - (k_2 + p_1)^2]^{a_2}} \ \times \frac{1}{[m^2 - (k_3 + p_1)^2]^{a_3} \, [m^2 - (k_3 + p_2)^2]^{a_4} \, [m^2 - (k_2 + p_2)^2]^{a_5}} \ \times \frac{1}{[m^2 - (k_1 + p_2)^2]^{a_6} \, [-k_1^2]^{a_7} \, [-(k_1 - k_2)^2]^{a_8} \, [-(k_2 - k_3)^2]^{a_9}} \ \times \frac{1}{[-(k_1 - k_3)^2]^{a_{10}} \, [-k_2^2]^{-a_{11}} \, [-k_3^2]^{-a_{12}}} \]

at \( p_1^2 = m^2, \ p_2^2 = m^2, \ q^2 = (p_1 - p_2)^2 = 4m^2. \)

Each index can be positive but the total number of positive indices cannot be more than 9. This family of integrals can be represented as the union of eight subfamilies.
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Evaluating planar three-loop vertex integrals at threshold

(1)  (2)  (3)  (4)

(5)  (6)  (7)  (8)
51 master integrals

\[ F_{0,0,0,1,1,1,0,0,0,0,0,0,0}, \ F_{0,0,0,0,1,1,0,0,0,1,0,1}, \ F_{0,0,0,0,1,1,0,0,0,1,1,0}, \ F_{0,0,1,0,0,0,0,0,1,0,1,1,0}, \]
\[ F_{0,0,1,0,0,1,0,1,1,0,0,0}, \ F_{0,0,1,0,0,1,0,1,2,0,0,0}, \ F_{0,0,1,0,1,1,0,0,0,1,0,0}, \ F_{0,0,0,0,0,1,0,1,0,1,1,1}, \]
\[ F_{0,0,1,0,0,1,0,1,0,1,1,1,0}, \ F_{0,0,1,0,0,1,0,1,0,2,0,0}, \ F_{0,0,1,0,0,1,0,2,1,0,1,0}, \ F_{0,0,1,0,0,1,1,1,0,0,1,0}, \]
\[ F_{0,0,1,0,1,1,0,0,1,1,0,0}, \ F_{0,0,1,0,1,1,0,0,1,1,2,0,0}, \ F_{0,0,1,0,1,1,0,0,1,1,2,0,0}, \ F_{0,0,1,0,1,2,0,1,0,1,1,0}, \]
\[ F_{0,0,1,0,1,2,0,0,1,1,1,0}, \ F_{0,0,1,1,0,1,0,1,1,2,0,0}, \ F_{0,0,1,1,0,1,1,1,1,0,0,0}, \ F_{0,1,1,0,0,1,0,1,0,1,0,1}, \]
\[ F_{0,1,1,0,0,1,0,1,2,0,1,0,1}, \ F_{0,1,1,0,1,1,0,0,1,1,1,0,0}, \ F_{0,1,1,0,1,1,0,0,1,1,0,0,0}, \ F_{0,1,1,0,1,1,0,0,1,1,0,0,0}, \]
\[ F_{0,0,1,0,1,1,0,1,0,1,1,1,1}, \ F_{0,0,2,0,1,1,0,1,0,1,1,1}, \ F_{0,0,1,1,1,1,0,1,0,1,1,0}, \ F_{0,0,1,1,1,1,0,1,0,1,2,0}, \]
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— Evaluating planar three-loop vertex integrals at threshold
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Evaluating planar three-loop vertex integrals at threshold

(16)  (17)  (18a)  (19)  (20)

(21)  (22)  (23a)  (24b)  (25)  (26)
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Evaluating planar three-loop vertex integrals at threshold

(42) (43a) (44b) (45c) (46) (47a) (48) (49) (50) (51)
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\frac{s}{m^2} = -\frac{(1 - x)^2}{x}
\]

The values \( x = 0 \) and \( x = -1 \) correspond to \( s = 0 \) and \( s = 4m^2 \).
Our goal are integrals at $s = q^2 \equiv (p_2 - p_2)^2 = 4m^2$.

Turn to the corresponding family of integrals at general $q^2$ and introduce

$$\frac{s}{m^2} = -\frac{(1 - x)^2}{x}$$

The values $x = 0$ and $x = -1$ correspond to $s = 0$ and $s = 4m^2$.

DE

$$f'(\epsilon, x) = \epsilon \tilde{A}'(x) f(x, \epsilon),$$

where $\tilde{A} = \sum_k \tilde{A}_{\alpha_k} \log(\alpha_k)$ and the letters $\alpha_k$ are $x, 1 + x, 1 - x, 1 + x + x^2$. 
90 elements of this basis \( f(x) \) are

\[
\{ F_{0,0,0,3,3,0,0,0,0,0,0,0}, \quad \varepsilon \frac{x^2 - 1}{x} F_{0,0,2,1,3,3,0,0,0,0,0,0}, \quad \cdots \\
\varepsilon^6 \frac{(1 - x^2)^2}{x^2} F_{1,0,1,1,1,1,1,1,0,0,0,0}, \quad (1 - 2\varepsilon)\varepsilon^4 F_{1,2,1,0,0,0,0,1,1,1,0,0,1} \}
\]
90 elements of this basis $f(x)$ are

$$\{ F_{0,0,0,3,3,0,0,0,0,0,0,0} , \quad \varepsilon \frac{x^2 - 1}{x} F_{0,0,2,1,3,3,0,0,0,0,0,0,0,0,0} , \ldots \}$$

$$\varepsilon^6 \left(1 - x^2 \right)^2 \frac{F_{1,0,1,1,1,1,1,1,1}}{x^2} , \quad (1 - 2\varepsilon)\varepsilon^4 F_{1,2,1,0,0,0,1,1,1,0,0,1}$$

A solution in an epsilon-expansion with coefficients written in terms of Goncharov (multiple) polylogarithms (GPL)

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

with indices $a_i$ taken from the seven-letters alphabet

$$\{0, r_1, r_3, -1, r_4, r_2, 1\}$$

with

$$r_{1,2} = \frac{1}{2} \left(1 \pm \sqrt{3} \, i \right) , \quad r_{3,4} = \frac{1}{2} \left(-1 \pm \sqrt{3} \, i \right) .$$
A typical expression for analytical results for the elements of the basis

\[ \text{ep}^{-4*(-24*G\{\{\{0\}, 1\}\}G\{0, -1\}, 1\} + \\
24*G\{0, -1\}, 1\}G\{0, -1\}, x\} - 23*G\{0, -1\}, 1\}G\{0, 0\}, x\} - \\
12*G\{\{\{0\}, 1\}\}G\{0, 1\}, 1\} + 12*G\{0, -1\}, 1\}G\{0, 1\}, 1\} - \\
(23*G\{0, 0\}, x\}G\{0, 1\}, 1\})/2 + 12*G\{0, -1\}, 1\}G\{0, 1\}, 1\} + \\
6*G\{0, 1\}, 1\}G\{0, 1\}, x\} + 12*G\{0, -1\}, 1\}G\{1, 0\}, x\} + \\
6*G\{0, 1\}, 1\}G\{1, 0\}, x\} - 9*G\{0, -1\}, 1\}G\{r1, 0\}, x\} - \\
9*G\{0, -1\}, 1\}G\{r2, 0\}, x\})/2 - 9*G\{0, 1\}, 1\}G\{r2, 0\}, x\} - \\
12*G\{\{\{0\}, 1\}\}G\{0, -1\}, 1\} + 12*G\{0, -1\}, 1\}G\{0, 0\}, 1\} + \\
24*G\{\{\{0\}, 1\}\}G\{0, 0\}, 1\} - 48*G\{0, 0\}, 0\}G\{0, 0, 1\}, 1\} + \\
18*G\{\{\{0\}, 1\}\}G\{0, 0\}, 1\} + 24*G\{\{\{0\}, 1\}\}G\{0, 0, 1\}, 1\} - \\
(57*G\{\{\{0\}, 1\}\}G\{0, 0\}, 1\})/2 + 24*G\{\{\{1\}\}G\{0, 0\}, 1\}\} + \\
(21*G\{\{\{r1\}\}G\{0, 0\}, 1\})/2 - (21*G\{\{\{r2\}\}G\{0, 0\}, 1\})/2 - \\
6*G\{0, 0\}, x\}G\{0, 1\}, 1\} - 24*G\{\{\{1\}\}G\{0, 0\}, 1\}\} + \\
36*G\{\{\{\{1\}\}G\{0, 0\}, 0\}, x\} - 24*G\{\{\{\{1\}\}G\{0, 0\}, 0\}, x\} + \\
24*G\{\{\{0\}, 0\}, 0\}, x\} + 2*G\{\{\{0\}, 0\}, 0\}, x\} + 12*G\{\{\{0\}, 0\}, 0\}, x\} - \\
23*G\{\{\{0\}, 0\}, 1\}, 1\})/2 + \\
12*G\{\{\{0\}, 1\}, 0\}, x\} + (11*G\{0, 1\}, x\})/2 + \\
6*G\{0, 0\}, 1\})/2 - 24*G\{\{0, 0\}, 0\}, x\} + 12*G\{\{0, 0\}, 0\}, x\} + \\
15*G\{\{\{0\}, 0\}, 0\}, x\} + 6*G\{\{\{0\}, 0\}, 0\}, x\} + 12*G\{\{\{0\}, 0\}, 0\}, x\} - \\
9*G\{\{\{r1\}, 0\}, 1\})/2 + (3*G\{\{\{r1\}, 1\}, 1\}, 0\})/2 - \\
9*G\{\{\{r2\}, 0\}, 1\})/2 + (3*G\{\{\{r2\}, 1\}, 0\})/2 + \\
(3*G\{0, x\})*Zeta[3])/2 - (3*G\{\{\{r1\}\}, x\})*Zeta[3])/2 - \\
(3*G\{\{\{r2\}\}, x\})*Zeta[3])/2 - \\
(3*(16*G\{\{\{0\}, 0\}, 1\})/2 + 8*G\{0, 1\}, 1\}G\{0, 1\}, 1\} + \ldots
Threshold expansion

\[ F(a_1, \ldots, a_{12}; q^2, m^2) \sim \sum_{n=n_0}^{\infty} \sum_{j=0}^{3} (4m^2 - q^2)^{n-j\epsilon} F_{n,j}(a_1, \ldots, a_{12}; q^2). \]
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Our goal are one-scale integrals \( F_{0,0}(a_1, \ldots, a_{12}; m^2) \) defined with \( q^2 \) set to \( 4m^2 \).
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Threshold expansion

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Expand ‘naively’ in \( x + 1 \) the corresponding integrals. Introduce one more (13th) index for the order of this derivative in \( s \), i.e. deal with the family

\[ F'(a_1, \ldots, a_{12}, a_{13}) = \left( \frac{\partial}{\partial s} \right)^{-a_{13}} F(a_1, \ldots, a_{12}) \bigg|_{s=4m^2}. \]
Using IBP relations for integrals at general $q$ and expanding all the terms naively in $q^2$ at $q^2 = 4m^2 \rightarrow 15$ IBP relations.
Using IBP relations for integrals at general $q$ and expanding all the terms naively in $q^2$ at $q^2 = 4m^2$ → 15 IBP relations.

A naive differentiation in $s$ of all the terms of the naive expansion [P.A. Baikov and V.A. Smirnov’2000] → one more relation.
Using IBP relations for integrals at general $q$ and expanding all the terms naively in $q^2$ at $q^2 = 4m^2 \rightarrow 15$ IBP relations.

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A naive differentiation in $s$ of all the terms of the naive expansion [P.A. Baikov and V.A. Smirnov'2000] \(\rightarrow\) one more relation.

Then $F'(a_1, \ldots, a_{12}, a_{13})$ are reduced to master integrals (with FIRE).

They are all with $a_{13} = 0$, i.e directly correspond to the 51 master threshold integrals.
Matching at threshold
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\[ x = y - 1, \quad y \to 0: \]

\[ f'(\epsilon, y) = \epsilon \frac{\tilde{A}'(y)}{y} f(\epsilon, y), \]

where \( \tilde{A}'(y) = A_0 + yA_1 + y^2A_2 + \ldots \).
Matching at threshold

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where \( \tilde{A}'(y) = A_0 + yA_1 + y^2A_2 + \ldots. \)

In the language of differential equations, the naive part of the expansion near \( y = 0 \) corresponds to zero eigenvalues of the matrix \( A_0 \) while eigenvalues proportional to \( \epsilon \) correspond to other contributions.
It is technically not so easy to obtain expansions near $y = 0$ of the elements of the basis in higher orders in $y$. 
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Construct a polynomial $P = 1 + \sum_{r=1} P_r y^r$ such that the DE for the function $g$ defined by $f = Pg$ takes the form $yg'(y) = A_0 g(y)$ (with $A_0$ is independent of $y$).
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Then the solution of this equation is just \( g = y^{A_0} g_0 \) with a boundary value \( g_0 \).

We implemented this algorithm and constructed \( P_r \) up to \( r = 5 \).
Equating the part of our analytic results for the basis without \( \log(x + 1) \) and the naive part of the threshold expansion expressed in terms of the 51 threshold \( M_l \).
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expressed in terms of the 51 threshold Ml.

Solving these equations \( \rightarrow \) coefficients of the epsilon 
expansion of the Ml up to some order written in terms of GPL 
\( G(a_1, \ldots, a_n; 1) \) with \( a_1 \neq 1 \) and \( a_i \) taken from the alphabet 
\( \{0, r_1, r_3, -1, r_4, r_2, 1\} \).
Equating the part of our analytic results for the basis without \( \log(x + 1) \) and the naive part of the threshold expansion expressed in terms of the 51 threshold \( \text{Ml} \).

Solving these equations \( \rightarrow \) coefficients of the epsilon expansion of the \( \text{Ml} \) up to some order written in terms of GPL \( G(a_1, \ldots, a_n; 1) \) with \( a_1 \neq 1 \) and \( a_i \) taken from the alphabet \( \{0, r_1, r_3, -1, r_4, r_2, 1\} \).

Examples of our results
[J. Henn, A. Smirnov and V. Smirnov’15]
\[ F_{0,0,1,0,1,1,0,1,0,1,1,1} = -\frac{27}{2} \log(2) G_R(0, 0, r_2, -1) - \frac{181\zeta(5)}{32} - \frac{21}{2} \log^2(2) \zeta(3) \]
\[ + \frac{115\pi^2 \zeta(3)}{48} - 12 \text{Li}_5 \left( \frac{1}{2} \right) - 12 \log(2) \text{Li}_4 \left( \frac{1}{2} \right) - \frac{2 \log^5(2)}{5} + \frac{1}{6} \pi^2 \log^3(2) \]
\[ - \frac{81}{8} G_R(0, 0, r_4, 1) \log(2) + \frac{277}{960} \pi^4 \log(2) , \]
\[ F_{0,0,1,1,1,1,0,1,0,1,1,0} = -\frac{27}{4} \log(2) G_R(0, 0, r_2, -1) - \frac{341\zeta(5)}{64} - \frac{21}{4} \log^2(2) \zeta(3) \]
\[ + \frac{211\pi^2 \zeta(3)}{96} - 6 \text{Li}_5 \left( \frac{1}{2} \right) - 6 \log(2) \text{Li}_4 \left( \frac{1}{2} \right) - \frac{\log^5(2)}{5} \]
\[ + \frac{1}{12} \pi^2 \log^3(2) - \frac{81}{16} G_R(0, 0, r_4, 1) \log(2) + \frac{277\pi^4 \log(2)}{1920} , \]
\[ F_{0,0,1,1,1,1,0,1,0,1,2,0} = -\frac{1}{24\epsilon^3} + \frac{1}{3\epsilon^2} - \frac{25\pi^2}{96\epsilon} - \frac{13}{6\epsilon} - \frac{97\zeta(3)}{24} \]
\[ - \pi^2 \log(2) + \frac{7\pi^2}{4} + \frac{40}{3} , \ldots \]
\[ G(a_1, \ldots, a_n; 1) = G_R(a_1, \ldots, a_n) + i \ G_I(a_1, \ldots, a_n) \]
$G(a_1, \ldots, a_n; 1) = G_R(a_1, \ldots, a_n) + i \, G_I(a_1, \ldots, a_n)$

$G(a_1, \ldots, a_n; 1)$ satisfy various relations.
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\end{equation}

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A linear basis in this set of constants up to weight 3 [D. Broadhurst’98] in terms of known transcendental numbers.
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Constants present in results for Feynman integrals up to
weight 5 were discussed in
[Fleischer and M. Kalmykov’99, Davydychev M. Kalmykov’00,
M. Kalmykov and B. Kniehl’10].
\[
G(a_1, \ldots, a_n; 1) = G_R(a_1, \ldots, a_n) + i \, G_I(a_1, \ldots, a_n)
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For example,
\[
G_I(r_2) = -\frac{\pi}{3}, \quad G_R(-1) = \log(2),
\]
\[
G_R(0, 0, 1) = -\zeta(3), \quad G_R(0, 0, 0, 1) = -\frac{\pi^4}{90},
\]
\[
G_R(0, 0, 0, 0, 1) = -\zeta(5),
\]
\[
G_R(0, 0, 1, 1, -1) = -2 \text{Li}_5 \left(\frac{1}{2}\right) - 2 \text{Li}_4 \left(\frac{1}{2}\right) \log(2) - \frac{\pi^2 \zeta(3)}{96}
\]
\[
+ \frac{151 \zeta(5)}{64} - \frac{\log^5(2)}{15} + \frac{1}{18} \pi^2 \log^3(2) - \frac{1}{96} \pi^4 \log(2).
\]
Shuffle relations

\[ G(a_1, \ldots, a_{n_1}; x) \, G(b_1, \ldots, b_{n_2}; x) = \sum_{c=a \cup b} G(c_1, \ldots, c_{n_1+n_2}; x), \]
Shuffle relations

\[ G(a_1, \ldots, a_{n_1}; x) \, G(b_1, \ldots, b_{n_2}; x) = \sum_{c=a \oplus b} G(c_1, \ldots, c_{n_1+n_2}; x), \]

Zhao’s conjecture [J. Zhao’07]: all independent (polynomial) relations among GPL at nth roots of unity are shuffle, stuffle, regularization, distributions relations and seeded relations, and lifted relations thereof.
In our case, $n = 6$, we used the first four of the above type of relations and the complex conjugation relations

$$G(a_1^*, \ldots, a_n^*; 1) = G(a_1, \ldots, a_n; 1)^*$$

with $r_1^* = r_2$, $r_3^* = r_4$. 
In our case, \( n = 6 \), we used the first four of the above type of relations and the complex conjugation relations

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G(a_1^*, \ldots, a_n^*; 1) = G(a_1, \ldots, a_n; 1)^*
\]

with \( r_1^* = r_2, r_3^* = r_4 \).

The total number of these five sets of relations grows fast when the weight is increased. At weight 6, we have 654452 equations for the real parts and 654937 equations for the imaginary parts of \( G(a_1, \ldots, a_n; 1) \).
We solved these relations up to weight 5 and hope to do this for weight 6.
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It turns out that the resulting constants, independent in the sense of these relations are still linear dependent, i.e. one can find additional relations for genuine constants of a given weight $G(a_1, \ldots, a_n; 1)$, i.e. linearly express them in terms of a smaller set of such constants and products of constants of lower weights.
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It turns out that the resulting constants, independent in the sense of these relations are still linear dependent, i.e. one can find additional relations for genuine constants of a given weight $G(a_1, \ldots, a_n; 1)$, i.e. linearly express them in terms of a smaller set of such constants and products of constants of lower weights.

We did this with experimental mathematics using the PSLQ algorithm [H.R.P. Ferguson, D.H. Bailey, and S. Arno] and ginac [C. Bauer, A. Frink and R. Kreckel] to evaluate GPLs with a big accuracy.
Our basis for the real parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

$$\{\text{GR}[0, 0, r_2, -1], \text{GR}[0, 0, r_4, 1], \text{GR}[r_2, 1, 1, -1],$$
$$\text{GR}[r_2, 1, 1, r_3], \text{GR}[r_2, 1, r_2, -1]\}$$

and 25 products of constants of lower weights

$$\{\text{GR}[-1]^4, \text{GI}[r_2]^2 \text{GR}[-1]^2, \text{GI}[r_2]^4, \text{GR}[-1]^3 \text{GR}[r_4],$$
$$\text{GI}[r_2]^2 \text{GR}[-1] \text{GR}[r_4], \text{GR}[-1]^2 \text{GR}[r_4]^2, \text{GI}[r_2]^2 \text{GR}[r_4]^2,$$
$$\text{GR}[-1] \text{GR}[r_4]^3, \text{GR}[r_4]^4, \text{GI}[r_2] \text{GI}[0, r_2] \text{GR}[-1],$$
$$\text{GI}[r_2]^2 \text{GR}[0, r_2] \text{GR}[r_4], \text{GI}[0, r_2]^2, \text{GR}[-1]^2 \text{GR}[r_2, -1],$$
$$\text{GI}[r_2]^2 \text{GR}[r_2, -1], \text{GR}[-1] \text{GR}[r_4] \text{GR}[r_2, -1], \text{GR}[r_4]^2 \text{GR}[r_2, -1],$$
$$\text{GR}[r_2, -1]^2, \text{GR}[-1] \text{GR}[0, 0, 1], \text{GR}[r_4] \text{GR}[0, 0, 1],$$
$$\text{GI}[r_2] \text{GI}[0, 1, r_4], \text{GI}[r_2] \text{GI}[0, r_2, -1], \text{GR}[-1] \text{GR}[r_2, 1, -1],$$
$$\text{GR}[r_4] \text{GR}[r_2, 1, -1], \text{GR}[-1] \text{GR}[r_2, 1, r_3], \text{GR}[r_4] \text{GR}[r_2, 1, r_3]\}$$
Our basis for the imaginary parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

\{GI[0, 0, 0, r2], GI[0, 1, 1, r4], GI[0, 1, r2, -1], GI[0, 1, r2, r3],
GI[0, r2, 1, -1]\}

and 20 products of constants of lower weights

\{GI[r2] GR[-1]^3, GI[r2]^3 GR[-1], GI[r2] GR[-1]^2 GR[r4],
GI[r2]^3 GR[r4], GI[r2] GR[-1] GR[r4]^2, GI[r2] GR[r4]^3,
GI[0, r2] GR[-1]^2, GI[r2]^2 GI[0, r2], GI[0, r2] GR[-1] GR[r4],
GI[0, r2] GR[r4]^2, GI[r2] GR[-1] GR[r2, -1],
GI[r2] GR[4] GR[r2, -1], GI[0, r2] GR[r2, -1], GI[r2] GR[0, 0, 1],
GI[0, 1, r4] GR[-1], GI[0, 1, r4] GR[r4], GI[0, r2, -1] GR[-1],
GI[0, r2, -1] GR[r4], GI[r2] GR[r2, 1, -1], GI[r2] GR[r2, 1, r3]\}
Our basis for the real parts of $G(a_1, \ldots, a_5; 1)$ consists of 13 constants of weight 5

\{GR[0, 0, 0, 1, 1], GR[0, 0, 1, 1, 1], GR[0, 0, 1, 1, r4], 
GR[0, 0, 1, r2, -1], GR[0, 0, 1, r2, r3], GR[0, 0, 1, r2, r4], 
GR[0, 0, r2, 1, -1], GR[r2, 1, 1, -1, r2], GR[r2, 1, 1, 1, -1], 
GR[r2, 1, 1, r3], GR[r2, 1, 1, r2, -1], GR[r2, 1, 1, r2, r3], 
GR[r2, 1, 1, r4, -1]\}

and 63 products of constants of lower weights

\{GR[-1]^5, GI[r2]^2 GR[-1]^3, GI[r2]^4 GR[-1], GR[-1]^4 GR[r4], 
GI[r2]^2 GR[-1]^2 GR[r4], GI[r2]^4 GR[r4], GR[-1]^3 GR[r4]^2, 
GR[-1] GR[r4]^4, GR[r4]^5, GI[r2] GI[0, r2] GR[-1]^2, 
GI[r2]^3 GI[0, r2], GI[r2] GI[0, r2] GR[-1] GR[r4], 
GI[r2] GI[0, r2] GR[r4]^2, GI[0, r2]^2 GR[-1], GI[0, r2]^2 GR[r4], 
GR[-1]^3 GR[r2, -1], GI[r2]^2 GR[-1] GR[r2, -1], 
GR[-1] GR[r4]^2 GR[r2, -1], GI[r4]^3 GR[r2, -1], 
GI[r2] GI[0, r2] GR[r2, -1], GR[-1] GR[r2, -1]^2, 
GR[r4] GR[r2, -1]^2, GR[-1]^2 GR[0, 0, 1], GI[r2]^2 GR[0, 0, 1], 
GR[-1] GR[r4] GR[0, 0, 1], GR[r4]^2 GR[0, 0, 1], 
GR[r2, -1] GR[0, 0, 1], GI[r2] GI[0, 1, r4] GR[-1], 
GI[r2] GI[0, 1, r4] GR[r4], GI[0, r2] GI[0, 1, r4], 
GI[r2] GI[0, r2, -1] GR[-1], GI[r2] GI[0, r2, -1] GR[r4], 
GI[0, r2] GI[0, r2, -1], GR[-1]^2 GR[r2, 1, -1], 
GI[r2]^2 GR[r2, 1, -1], GR[-1] GR[r4] GR[r2, 1, -1], 
GR[r4]^2 GR[r2, 1, -1], GR[r2, 1, -1] GR[r2, 1, -1], 
GR[-1]^2 GR[r2, 1, r3], GI[r2]^2 GR[r2, 1, r3], 
GR[-1] GR[r4] GR[r2, 1, r3], GR[r4]^2 GR[r2, 1, r3], 
GR[r2, -1] GR[r2, 1, r3], GI[r2] GI[0, 0, 0, r2], 
GR[-1] GR[0, 0, r2, -1], GR[r4] GR[0, 0, r2, -1], 
GR[-1] GR[0, 0, r4, 1], GR[r4] GR[0, 0, r4, 1], 
GI[r2] GI[0, 1, 1, r4], GI[r2] GI[0, 1, r2, -1], 
GI[r2] GI[0, 1, r2, r3], GI[r2] GI[0, r2, 1, -1], 
GR[-1] GR[r2, 1, 1, -1], GR[r4] GR[r2, 1, 1, -1], 
GR[-1] GR[r2, 1, 1, r3], GR[r4] GR[r2, 1, 1, r3], 
GR[-1] GR[r2, 1, r2, -1], GR[r4] GR[r2, 1, r2, -1]\}
Our basis for the imaginary parts of $G(a_1, \ldots, a_5; 1)$ consists of 11 constants of weight 5

\{GI[0, 0, 0, 1, r2], GI[0, 0, 0, 1, r4], GI[0, 0, 0, r2, -1],
GI[0, 1, 1, -1, r2], GI[0, 1, 1, -1, r4], GI[0, 1, 1, r4],
GI[0, 1, r2, r3], GI[0, 1, r4, -1], GI[0, 1, r4, r1],
GI[0, r2, r3, r2], GI[0, r2, 1, 1, -1]\}

and 57 products of constants of lower weights

GI[r2]^{-3} GR[-1]^{-2}, GI[r2]^{-2} GR[-1]^{-2} GR[r4],
GI[r2]^{-2} GR[-1]^{-2} GR[r4], GI[0, r2] GR[-1]^{-2} GR[r4],
GI[r2]^{-2} GR[0, r2] GR[r4], GI[0, r2] GR[-1]^{-2} GR[r4],
GI[0, r2] GR[r4]^{-3}, GI[0, r2] GR[r4]^{-2}, GI[r2] GR[-1]^{-2} GR[r2, -1],
GI[r2]^{-3} GR[r2, -1], GI[0, r2] GR[-1]^{-1} GR[r4] GR[r2, -1],
GI[r2] GR[r4]^{-2} GR[r2, -1], GI[0, r2] GR[-1]^{-1} GR[r2, -1],
GI[0, r2] GR[r4] GR[r2, -1], GI[0, r2] GR[r2, -1]^{-2},
GI[r2] GR[-1]^{-1} GR[0, 0, 1], GI[r2] GR[r4] GR[0, 0, 1],
GI[0, r2] GR[0, 0, 1], GI[0, 1, r4] GR[-1]^{-2}, GI[r2]^{-2} GI[0, 1, r4],
GI[0, 1, r4] GR[-1]^{-1} GR[r4], GI[0, 1, r4] GR[r4]^{-2},
GI[0, 1, r4] GR[r2, -1], GI[0, r2, -1] GR[-1]^{-2},
GI[r2]^{-2} GI[0, r2, -1], GI[0, r2, -1] GR[-1]^{-1} GR[r4],
GI[0, r2, -1]^{-1} GR[r4]^{-2}, GI[0, r2, -1]^{-1} GR[r2, -1],
GI[r2] GR[-1]^{-1} GR[r2, 1, -1], GI[r2] GR[r4] GR[r2, 1, -1],
GI[0, r2] GR[r2, 1, -1], GI[r2] GR[-1]^{-1} GR[r2, 1, r3],
GI[r2] GR[r4] GR[r2, 1, r3], GI[0, r2] GR[r2, 1, r3],
GI[0, 0, 0, r2] GR[-1], GI[0, 0, 0, r2] GR[r4],
GI[r2] GR[0, 0, r2, -1], GI[r2] GR[0, 0, r4, 1],
GI[0, 1, 1, r4] GR[-1], GI[0, 1, 1, r4] GR[r4],
GI[0, 1, r2, -1] GR[-1], GI[0, 1, r2, -1] GR[r4],
GI[0, 1, r2, 3] GR[-1], GI[0, 1, r2, 3] GR[r4],
GI[0, r2, 1, -1] GR[-1], GI[0, r2, 1, -1] GR[r4],
GI[r2] GR[r2, 1, -1], GI[r2] GR[r2, 1, r3],
GI[0, r2, 1, r3], GI[0, r2, 1, r3]
\}
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*to be continued*
Evaluating Feynman integrals by uniformly transcendental differential equations

Conclusion

BACKUP SLIDES
Lee [R.N. Lee’14]: transition to UT basis in three steps (in the case of one variable).
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- Reduction to a Fuchsian form where the singularities at all the points $x^{(k)}$ (including $x = \infty$) are simple poles.
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- Reduction to a Fuchsian form where the singularities at all the points $x^{(k)}$ (including $x = \infty$) are simple poles.
- Normalizing eigenvalues of the matrices which are coefficients the Fuchsian singularities when one tries to make them proportional to $\epsilon$.
- Providing a linear dependence on $\epsilon$. 
At each of the three steps, one is looking for a proper linear transformation of the current basis. The first two steps are based on the so-called balance transformation

$$\mathcal{B}(\mathbb{P}, x_1, x_2|x) = \mathbb{P} + c \frac{x - x_2}{x - x_1} \mathbb{P},$$

where \( c \) is a constant, \( \mathbb{P} \), \( \mathbb{P} \) are the two complementary projectors, i.e. \( \mathbb{P}^2 = \mathbb{P} \) and \( \mathbb{P} = \mathbb{I} - \mathbb{P} \).
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\[ B(\mathbb{P}, x_1, x_2|x) = \mathbb{P} + c \frac{x - x_2}{x - x_1} \mathbb{P}, \]

where \( c \) is a constant, \( \mathbb{P}, \overline{\mathbb{P}} \) are the two complementary projectors, i.e. \( \mathbb{P}^2 = \mathbb{P} \) and \( \overline{\mathbb{P}} = I - \mathbb{P} \).

The idea of using a balance transformation is that with its help one can take care of one singular point \( x_1 \) (by providing a Fuchsian singularity or by normalizing eigenvalues corresponding to a given singular point) and not to spoil these properties at a second singular point, \( x_2 \).