Radcor-Loopfest, UCLA, June 18, 2015

Singularity structure of maximally supersymmetric scattering amplitudes

Jaroslav Trnka

California Institute of Technology

Nima Arkani-Hamed Jacob Bourjaily Freddy Cachazo Andrew Hodges	Zvi Bern Enrico Herrmann Sean Litsey James Stankowicz	Lance Dixon Matt von Hippel Andrew McLeod
1212.5605, 1312.2007, 1410.0354, 1412.8478	1412.8584, in progress	in progress

Radcor-Loopfest, UCLA, June 18, 2015

Scattering Inequalities

Jaroslav Trnka

California Institute of Technology

Nima Arkani-Hamed
Jacob Bourjaily
Freddy Cachazo
Andrew Hodges

1212.5605, 1312.2007, 1410.0354, 1412.8478

Zvi Bern
Enrico Herrmann
Sean Litsey
James Stankowicz

1412.8584, in progress

Lance Dixon
Matt von Hippel
Andrew McLeod

in progress

Plan of the talk

- Integrand of scattering amplitudes
- Planar N=4 SYM: Amplituhedron
- Non-planar extension
- Positivity of ratio function
- N=8 SUGRA and poles at infinity

Scattering Amplitudes

- Basic objects in Quantum Field Theory
- Predictions for colliders: cross-sections
- My motivation: new ideas in QFT
  - Find hidden properties of amplitudes
  - Exploit them in new methods of calculation
  - Generalize to other cases

Integrand

- Finite well-defined rational function before integration

$I(\ell_j, k_i, s_i)$ sum of Feynman diagrams

$$\Omega = d^4\ell_1 \ldots d^4\ell_L \, I(\ell_j, k_i, s_i) \qquad A = \int_{\ell_j \in \mathbb{R}} \Omega$$

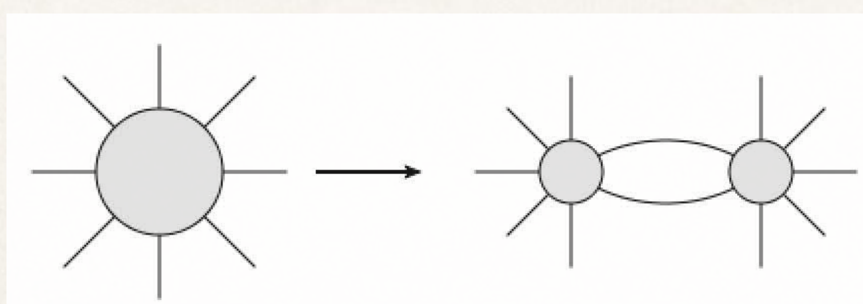
- Qualitative information about the final amplitudes
  - Collinear limits: IR divergencies
  - Poles at infinity: UV structure
  - Types of singularities: transcendental properties

Integrand

- Fixed by principles of locality and unitarity
- Re-express the integrand in the basis of integrals

$$I = \sum_j c_j \, I_j$$

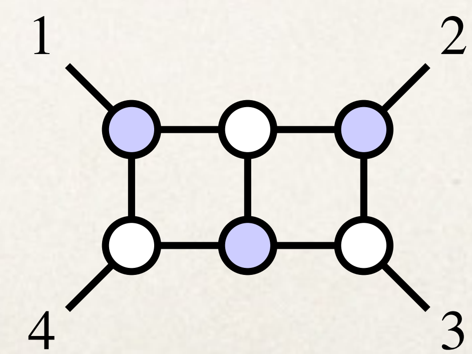
- Fix coefficients using cuts



$$\ell^2 = (\ell + Q)^2 = 0$$

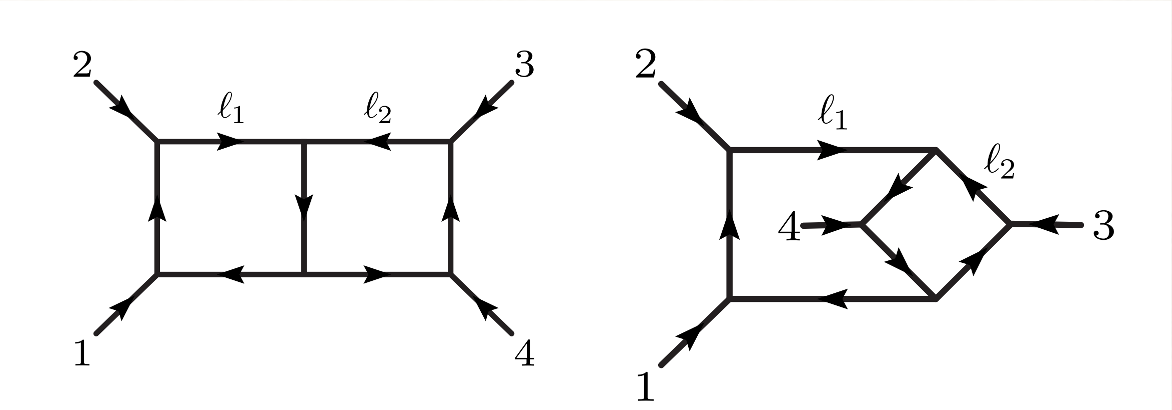
- Maximal cuts, leading singularities:

  (Bern, Dixon, Kosower)

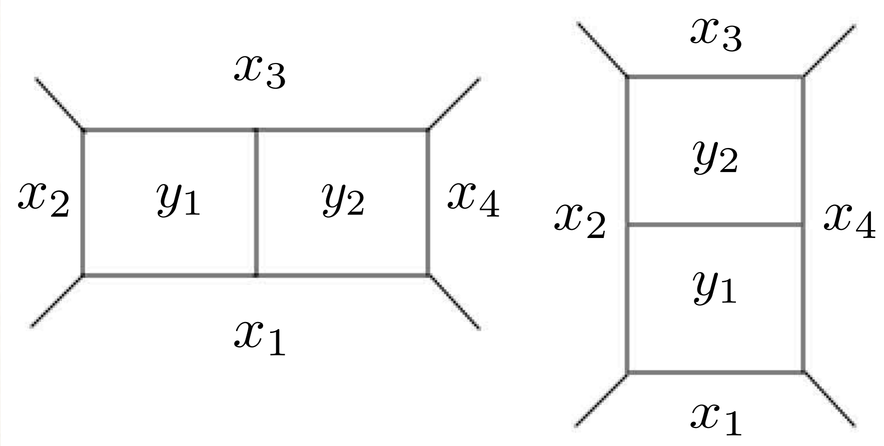


Planar limit

- The integrand defined as a sum of diagrams
  - No global loop momenta
  - Each diagram: its own labels



- Planar limit: dual variables



$$k_1 = (x_1 - x_2) \quad k_2 = (x_2 - x_3)$$
$$\ell_1 = (x_3 - y_1) \quad \ell_2 = (y_2 - x_3)$$
etc

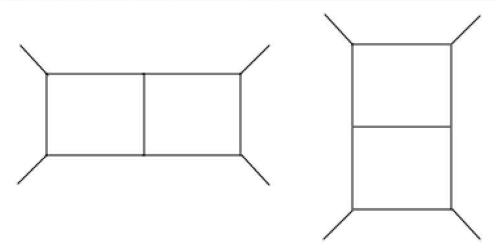
Global labels

Integrand: <u>single</u> function

Conditions on the amplitude

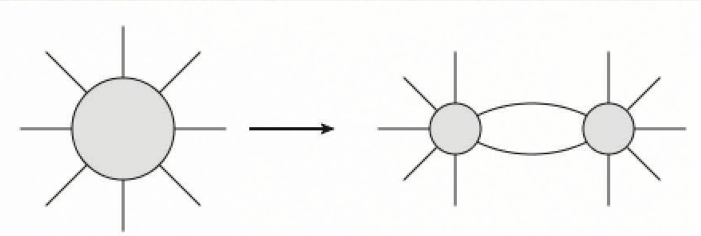
Standard methods

- Planar diagrams

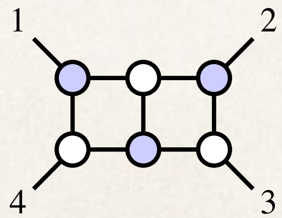


Locality + Planarity

- Match physical cuts/singularities



Unitarity

$\mathrm{Cut}(I) =$ 

Construction not known in general

Alternative

- Same set of conditions
- Packaged in a different way

?

Complete set known

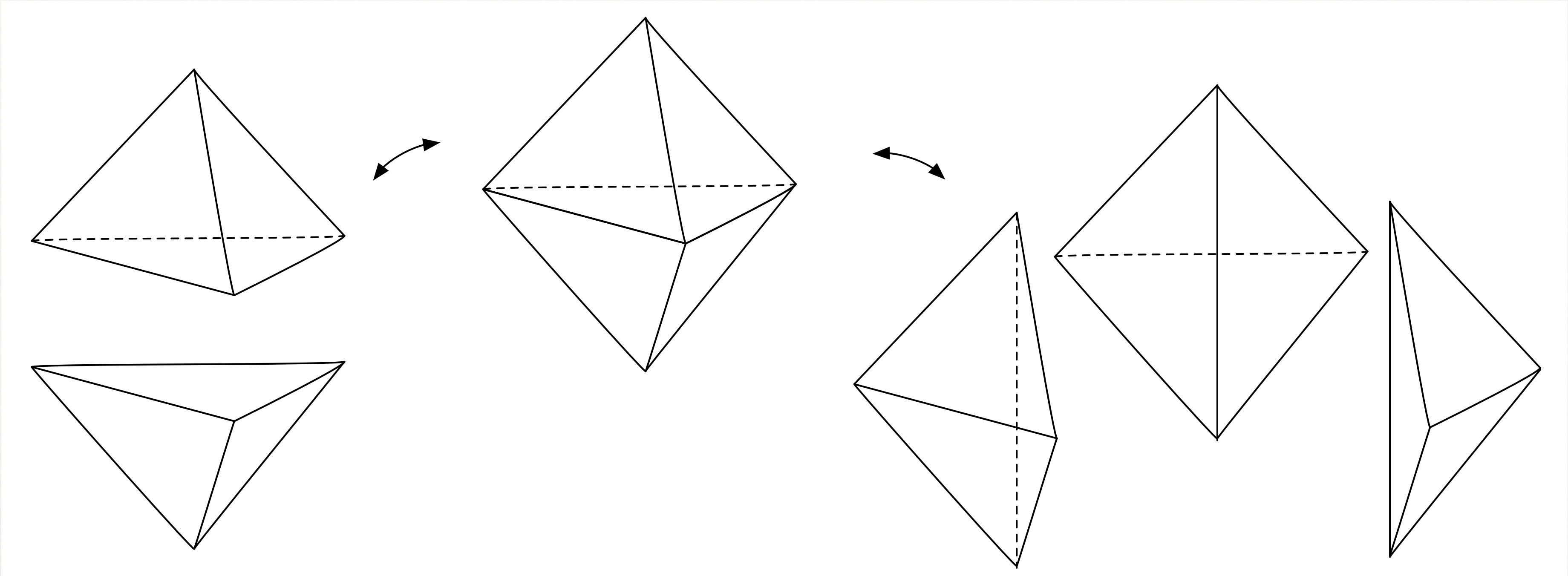
Toy model: N=4 SYM

- “Simplest Quantum Field Theory”
- Toy model for QCD
  - Tree-level amplitudes identical
  - Loop amplitudes simpler (results up to 7-loops)
- Planar: conformal + dual conformal, convergent series
- Past: new methods for amplitudes originated here

Volume of polyhedron

(Hodges 2009)

- New kinematical variables — momentum twistors
  $Z \in \mathbb{C}^3$
- Tree-level process: $gg \to 5g$
- Comparison of two calculations of recursion relations



(Picture by Stavros Garoufalidis)

The Amplituhedron

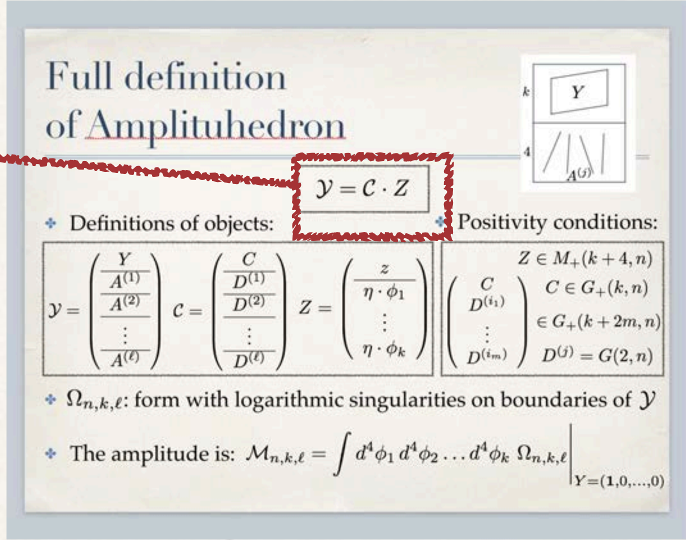
(Arkani-Hamed, JT 2013)

- Generalization of polyhedra to Grassmannian

$$\boxed{\mathcal{Y} = \mathcal{C}\cdot Z}$$

Amplituhedron

Positive matrices: Minors are positive $\begin{vmatrix} * & * \\ * & * \end{vmatrix} > 0$



Full definition of Amplituhedron

$\boxed{\mathcal{Y}=\mathcal{C}\cdot Z}$

- Definitions of objects:

$$\mathcal{Y}=\begin{pmatrix} Y \\ A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(\ell)} \end{pmatrix} \quad \mathcal{C}=\begin{pmatrix} C \\ D^{(1)} \\ D^{(2)} \\ \vdots \\ D^{(\ell)} \end{pmatrix} \quad \mathcal{Z}=\begin{pmatrix} z \\ \eta\cdot\phi_1 \\ \vdots \\ \eta\cdot\phi_k \end{pmatrix}$$

- Positivity conditions:

$$Z \in M_+(k+4,n) \quad C \in G_+(k,n) \quad \begin{pmatrix} C \\ D^{(i_1)} \\ \vdots \\ D^{(i_m)} \end{pmatrix} \in G_+(k+2m,n) \quad D^{(j)} = G(2,n)$$

- $\Omega_{n,k,\ell}$: form with logarithmic singularities on boundaries of $\mathcal{Y}$
- The amplitude is: $\mathcal{M}_{n,k,\ell} = \int d^4\phi_1\, d^4\phi_2 \ldots d^4\phi_k\; \Omega_{n,k,\ell}\Big|_{Y=(1,0,\ldots,0)}$

- Integrand in planar N=4 SYM: volume of this space
  - Geometry labeled by three labels
  - Derivation: locality and unitarity
  - Check against reference data

n number of particles

k helicity index

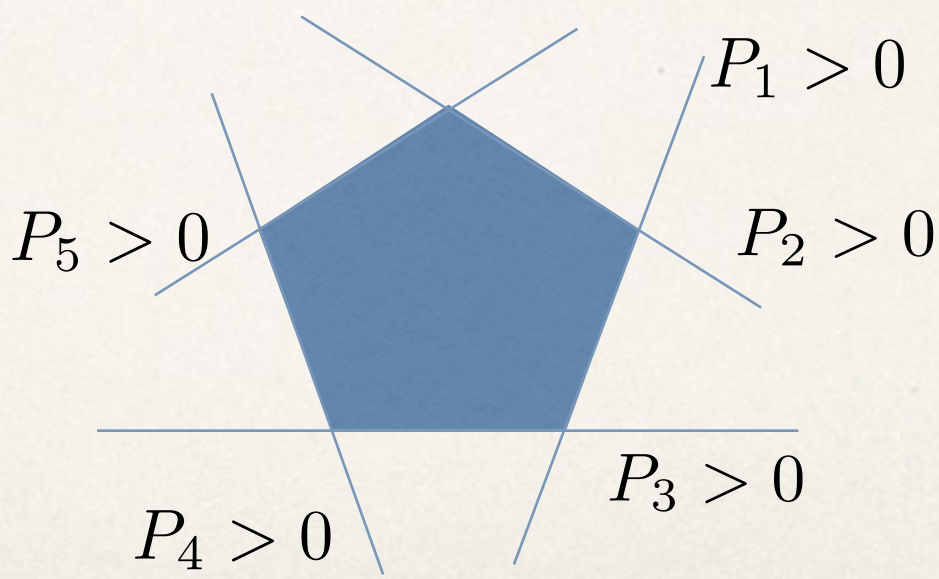
ℓ number of loops

Inequalities

- Volume: logarithmic form $\Omega \sim \dfrac{dx}{x}$ near $x = 0$
- Amplituhedron variables z_i

$$(p_i, \epsilon_j, \ell_k) \to (x_i, \tilde{\eta}_j, y_k) \to (Z_i, \eta_j, Z^{(k)}_{AB}) \to z_i$$

- Inequalities $P_j(z_i) \geq 0$

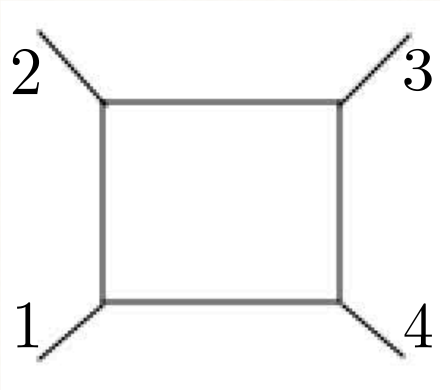


Legal and illegal boundaries

- Cuts of the amplitude: localize z_i
- Inequalities hold $P_j(z_i) \geq 0$ $\qquad \ell_k \in \mathbb{C} \leftrightarrow z_i > 0$
  - Point inside the Amplituhedron space
  - Physical cut or singularity of the amplitude
- One or more inequalities violated $P_j(z_i) < 0$
  - Point outside the Amplituhedron space
  - Unphysical cut or singularity of the amplitude

Example 1: One-loop amplitude

- Consider 4pt one-loop amplitude

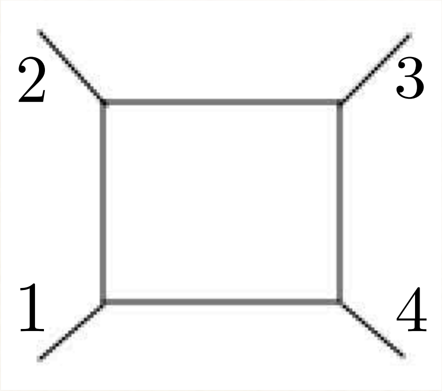


- Inequalities: $z_1, z_2, z_3, z_4 \geq 0$
- Boundaries of the space: $z_1, z_2, z_3, z_4 = (0, \infty)$
- Differential form

$$\Omega = \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3} \frac{dz_4}{z_4}$$

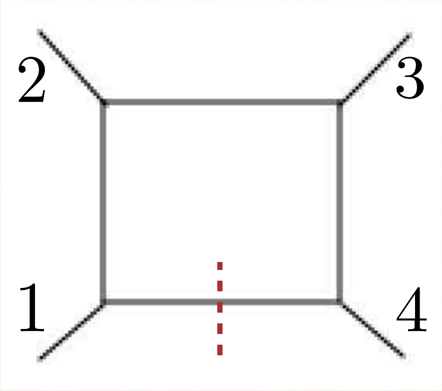
Example 1: One-loop amplitude

- Cuts of the amplitude



Example 1: One-loop amplitude

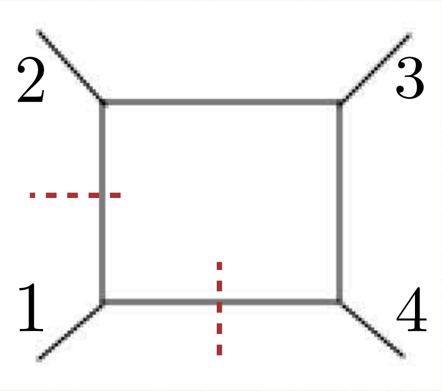
- Cuts of the amplitude



$$z_1 = 0$$

Example 1: One-loop amplitude

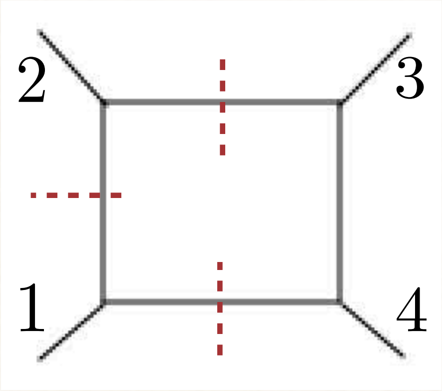
- Cuts of the amplitude



$$z_1 = 0$$
$$z_2 = 0$$

Example 1: One-loop amplitude

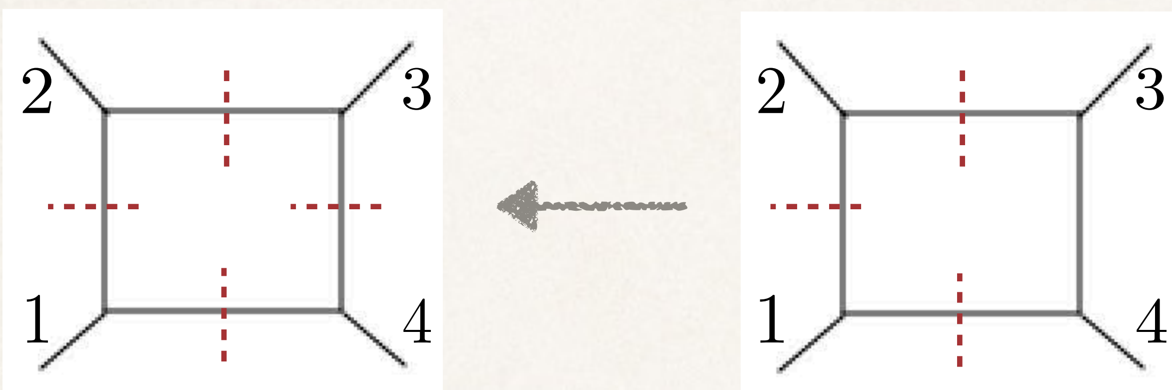
- Cuts of the amplitude



$$z_1 = 0$$
$$z_2 = 0$$
$$z_3 = 0$$

Example 1: One-loop amplitude

- Cuts of the amplitude



$$z_4 = 0$$

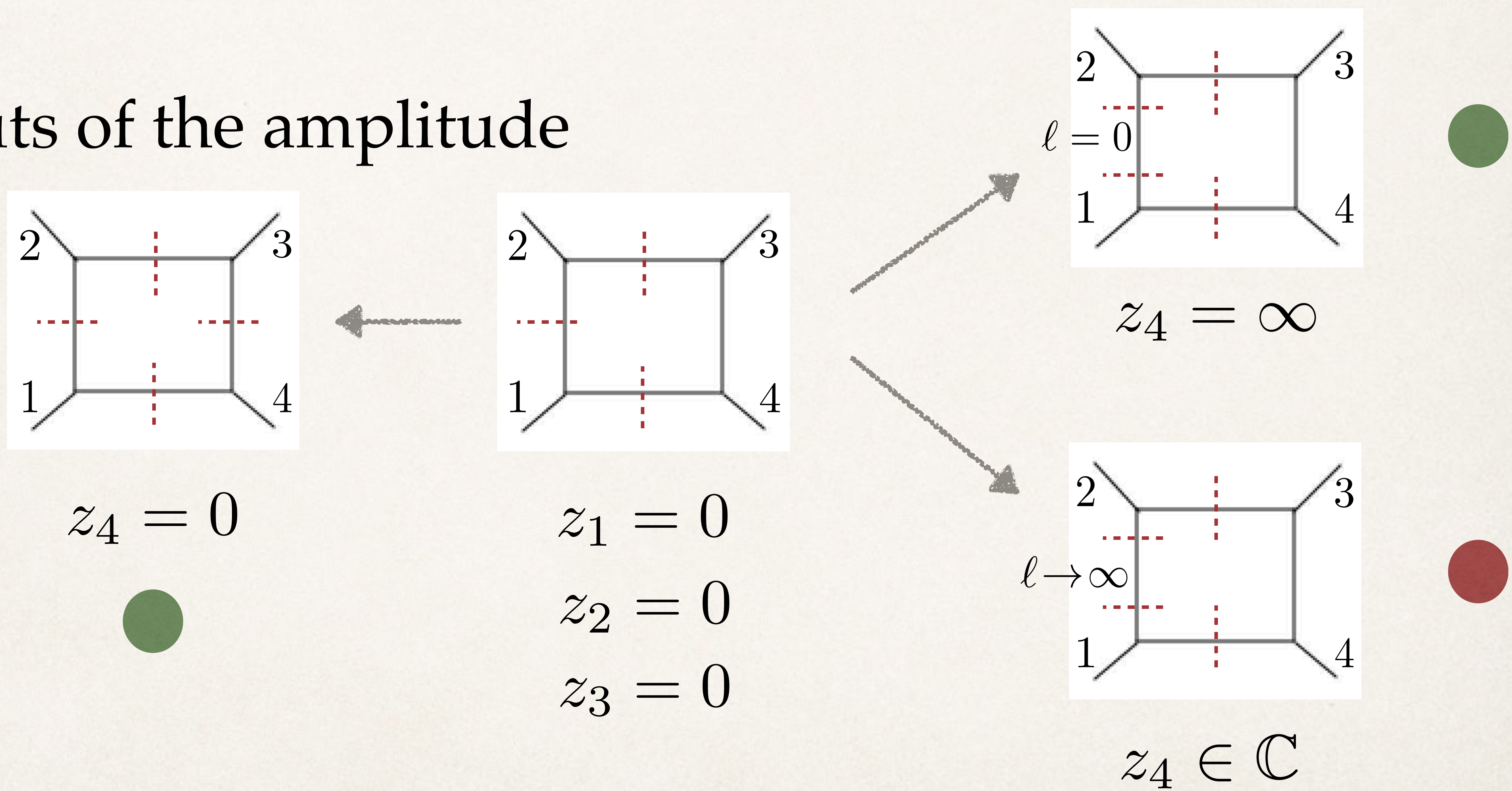
$$z_1 = 0$$

$$z_2 = 0$$

$$z_3 = 0$$

Example 1: One-loop amplitude

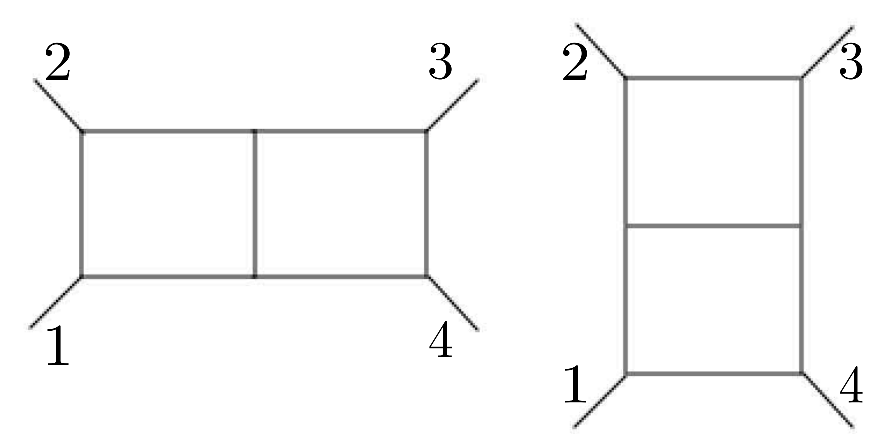
- Cuts of the amplitude



"no-triangle"

Example 2: Two-loop amplitude

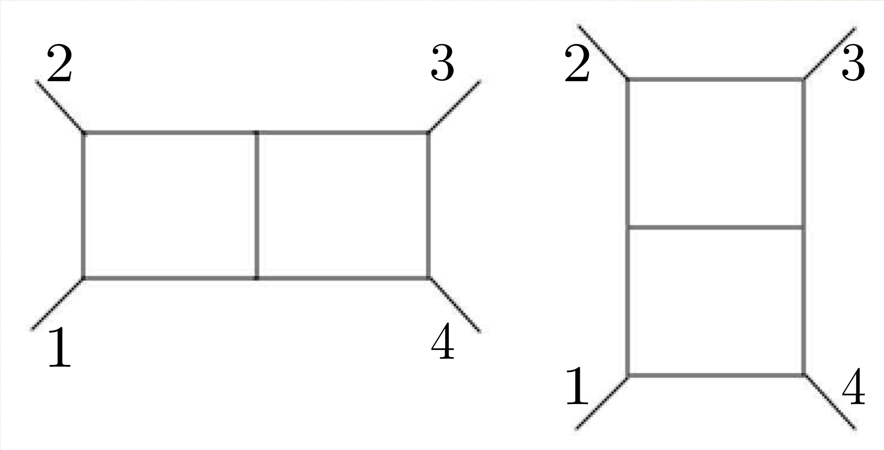
- Consider 4pt two-loop amplitude



- Inequalities: $$z_1, z_2, z_3, z_4 \geq 0$$
$$z_5, z_6, z_7, z_8 \geq 0$$
$$(z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0$$

Example 2: Two-loop amplitude

- Consider 4pt two-loop amplitude

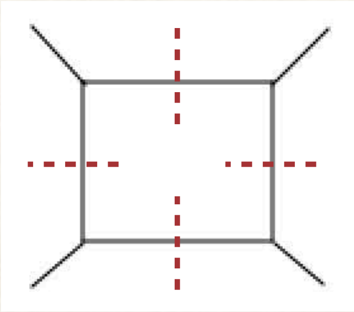


- Inequalities: $z_1, z_2, z_3, z_4 \geq 0$

$$z_5, z_6, z_7, z_8 \geq 0$$

$$(z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0$$

- Check: one-loop cut



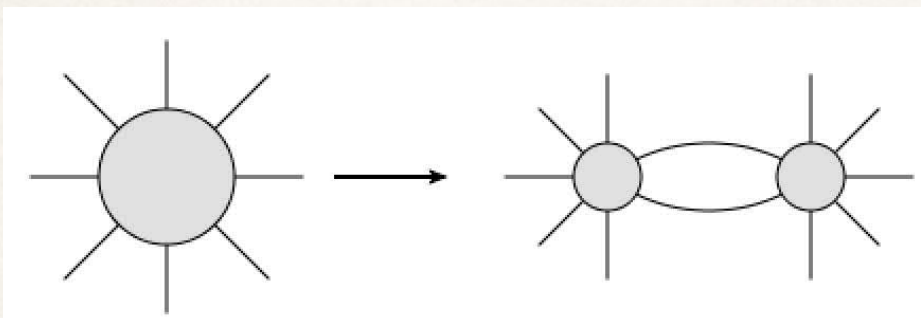
$$z_1 = 0$$
$$z_2 = 0$$
$$z_3 = 0$$
$$z_4 = 0$$

$$-z_5 z_6 - z_7 z_8 \geq 0$$

Ω vanishes on this cut

Example 3: Unitarity cut

- Standard formulation



$$\text{Cut}\, M_{n,\ell} = \sum_{\ell_1+\ell_2=\ell-1} M_{n_1,\ell_1} M_{n_2,\ell_2}$$

- Set of inequalities split into two sets

$$P_j(z_i) \geq 0 \xrightarrow{z_1 = z_2 = 0} \begin{array}{ll} P_j^{(1)}(z_i) \geq 0 & i = 3, \ldots, k \\ P_j^{(2)}(z_i) \geq 0 & i = k+1, \ldots, m \end{array}$$

$i = 1, \ldots, m$

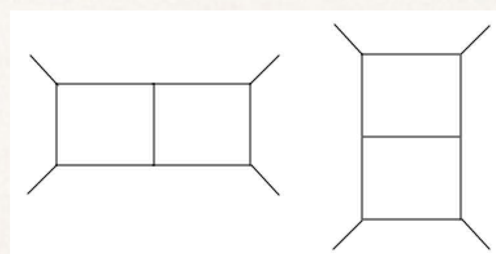
where k is a free parameter

Physics vs geometry

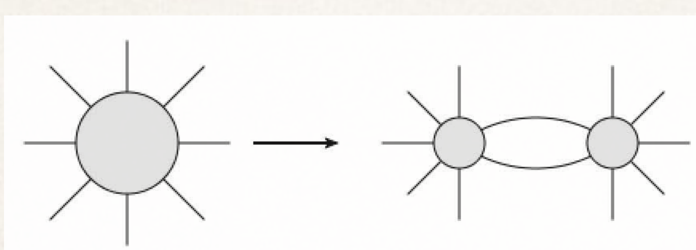
Standard methods

- Planar diagrams

Locality + Planarity

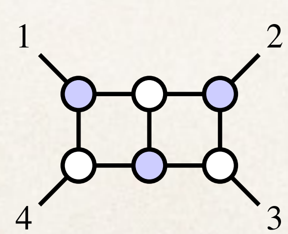


- Match physical cuts/singularities



Unitarity

$$\mathrm{Cut}(I) =$$



Construction not known in general

Amplituhedron

- Inequalities

$$P_j(z_i) \geq 0$$

- Logarithmic form

$$\Omega \sim \frac{dx}{x}$$

<u>Complete set known</u>

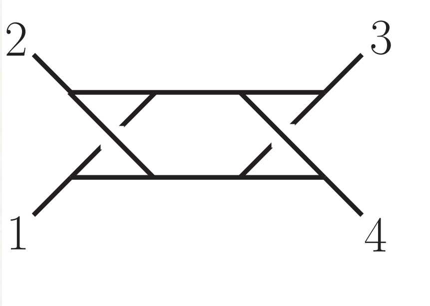
Non-planar amplitudes

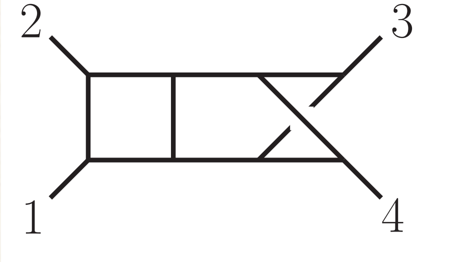
- No global variables: standard k_i, ℓ_k
- No single form, sum of diagrams

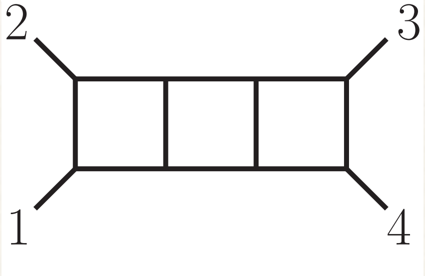
$$\Omega = \sum_{\sigma,j} C_j \cdot \Omega_j(k_i, \ell_k)$$

C_j color factor

- Each has its own variables



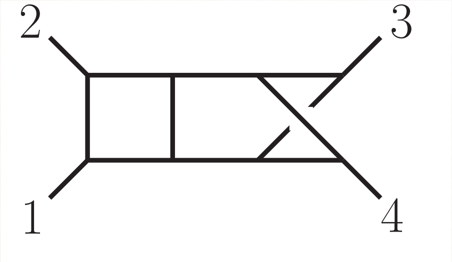




etc

Constraints

- Inspired by the planar sector we conjecture:
  - Logarithmic singularities $\Omega \sim \dfrac{dx}{x}$
  - No poles at $\ell \to \infty$
- Stronger condition: each diagram individually



$$I_j(k_i, \ell_k) = \frac{N_j(k_i, \ell_k)}{P_1^2 P_2^2 \ldots P_m^2}$$

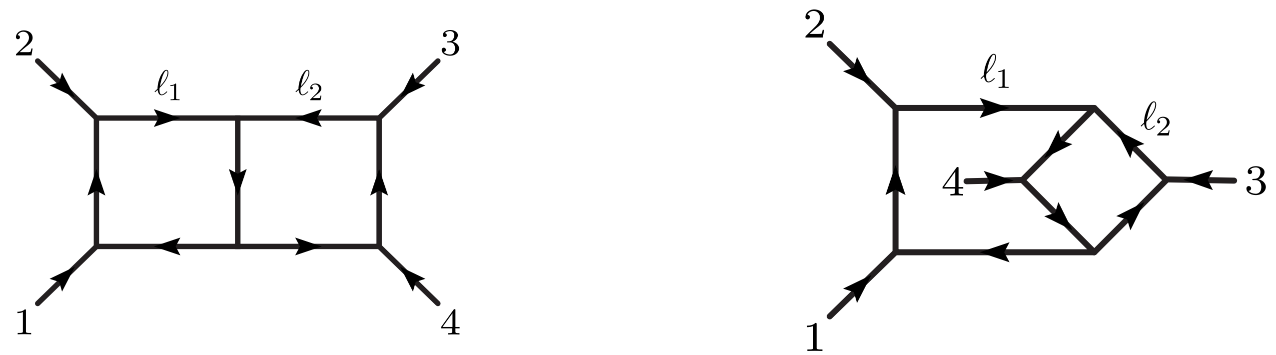
- Find the basis and expand the amplitude

Evidence 1: Two-loop amplitude

- Expansion of the 4pt two-loop amplitude

(Bern, Rozowsky, Yan 1997)

Two basis integrals



$$N_1 = (k_1 + k_2)^2 \qquad N_2 = (k_1 + k_2)^2$$

Double Poles

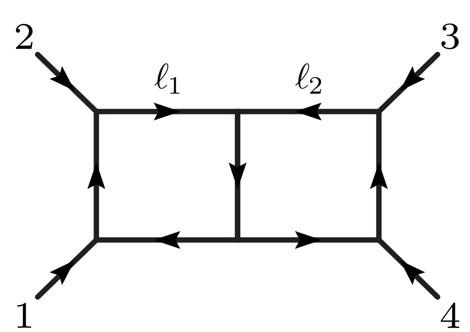
Poles at infinity

Evidence 1: Two-loop amplitude

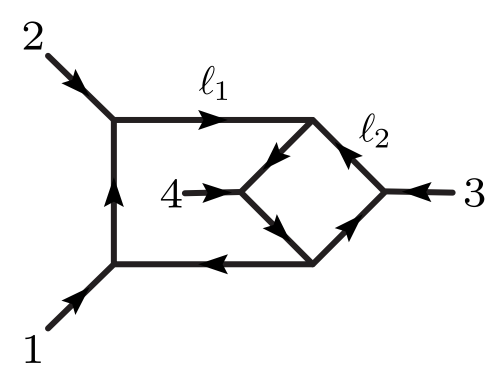
- Expansion of the 4pt two-loop amplitude

(Bern, Rozowsky, Yan 1997)

Two basis integrals



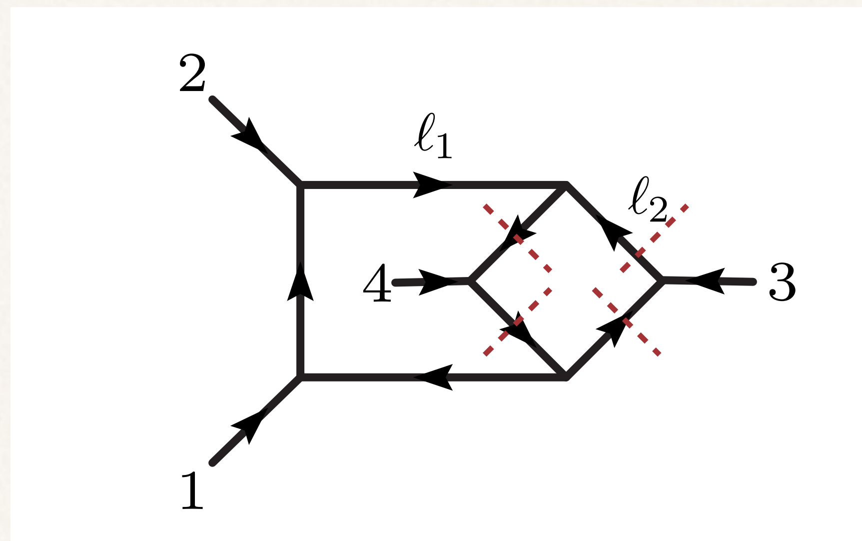
$$N_1 = (k_1 + k_2)^2$$



$$N_2 = (k_1 + k_2)^2$$

Double Poles	NO	YES
Poles at infinity	NO	YES

Evidence 1: Two-loop amplitude

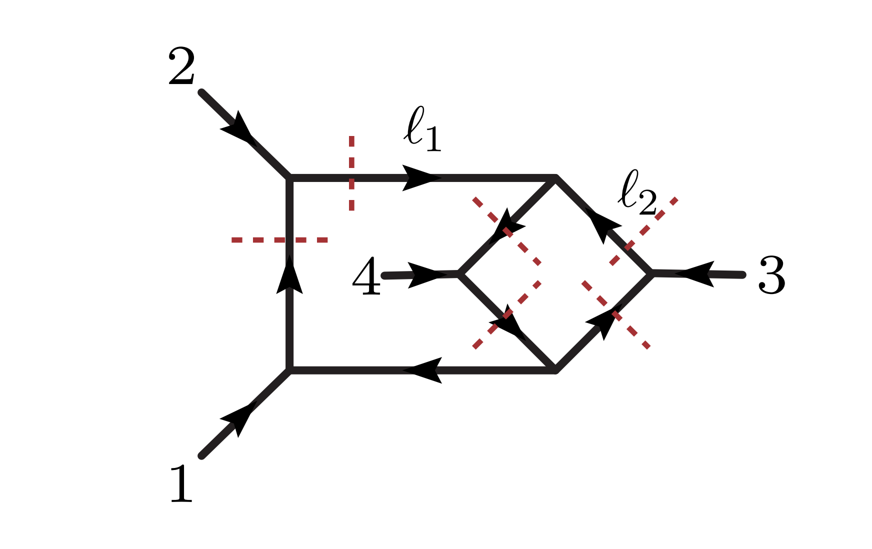


$$dI = \frac{d^4\ell_1\, d^4\ell_2\ (p_1+p_2)^2}{\ell_1^2(\ell_1-k_2)^2(\ell_1-k_1-k_2)^2\ell_2^2(\ell_2-k_3)^2(\ell_1+\ell_2)^2(\ell_1+\ell_2+k_4)^2}$$

Perform cuts $\quad \ell_2^2 = (\ell_2-k_3)^2 = (\ell_1+\ell_2)^2 = (\ell_1+\ell_2+k_4)^2 = 0$

Localize ℓ_2 completely

Evidence 1: Two-loop amplitude

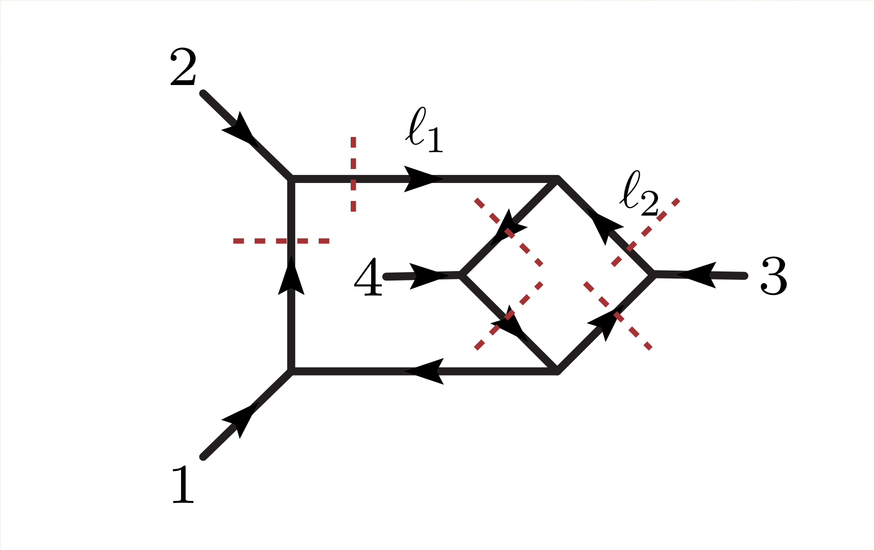


$$\mathrm{Cut}_1\, dI = \frac{d^4\ell_1}{\ell_1^2(\ell_1-k_2)^2(\ell_1-k_1-k_2)^2[(\ell_1+k_3)^2(\ell_1+k_4)^2-\ell_1^2(\ell_1+k_3+k_4)^2]}$$

Localize $\ell_1 = \alpha k_2$ by cutting $\ell_1^2 = (\ell_1 - k_2)^2 = 0$

and the Jacobian

Evidence 1: Two-loop amplitude

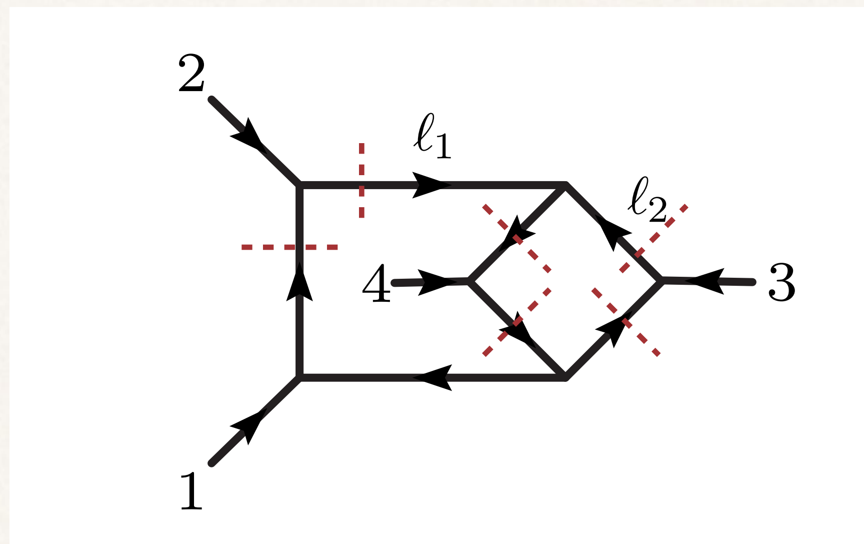


$$\mathrm{Cut}_{1,2}\, dI = \frac{d\alpha}{(\alpha+1)\alpha^2 tu}$$

Double pole for $\alpha = 0$

- There is also pole at infinity
- We want to find a numerator which cancels all that

Evidence 1: Two-loop amplitude



$$\mathrm{Cut}_{1,2}\, dI = \frac{d\alpha}{(\alpha+1)\alpha^2 tu}$$

Double pole for $\alpha = 0$

<u>New numerator</u>

$$\boxed{N = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2}$$

Cancels double pole

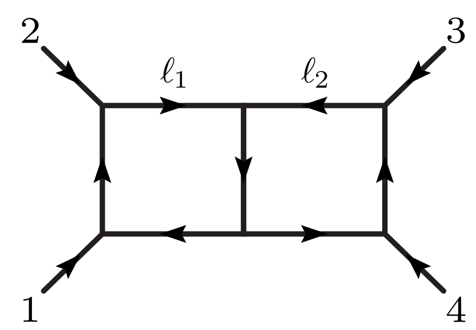
$$N \to \alpha s$$

Evidence 1: Two-loop amplitude

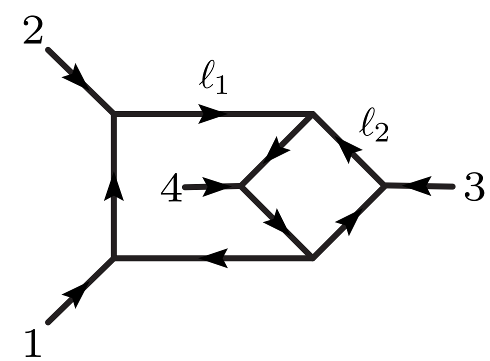
- New expansion of the 4pt two-loop amplitude

(Arkani-Hamed, Bourjaily, Cachazo, JT, 2014)

Two basis integrals



$N_1 = (k_1 + k_2)^2$



$N_2 = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2$

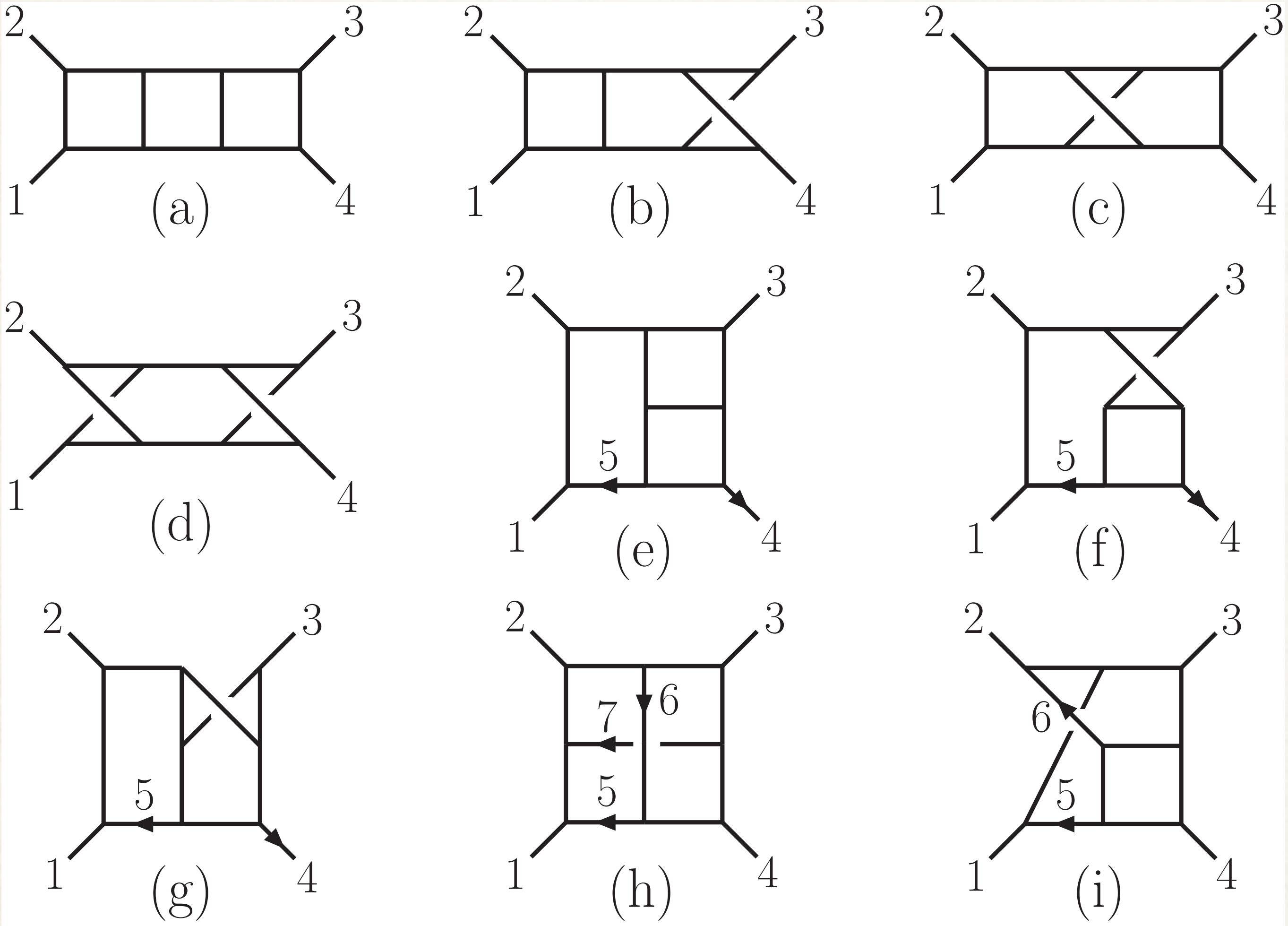
Double Poles	NO	NO
Poles at infinity	NO	NO

Expand amplitude in the basis: YES

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

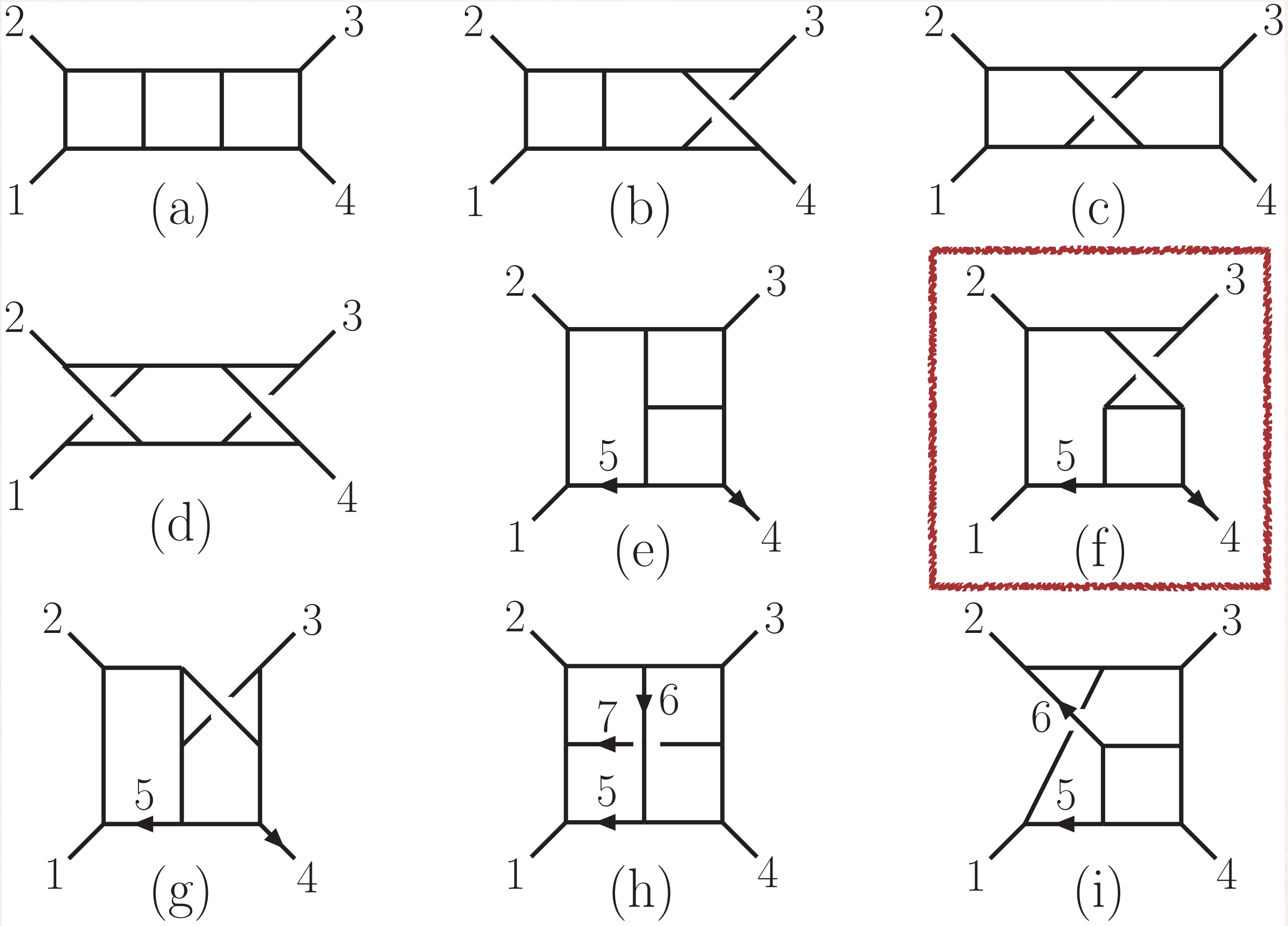
(Bern, Carrasco, Dixon, Johansson, Kosower 2007)



Numerator	Double pole	Pole at infinity
Original	YES	YES
BCJ	YES	YES

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

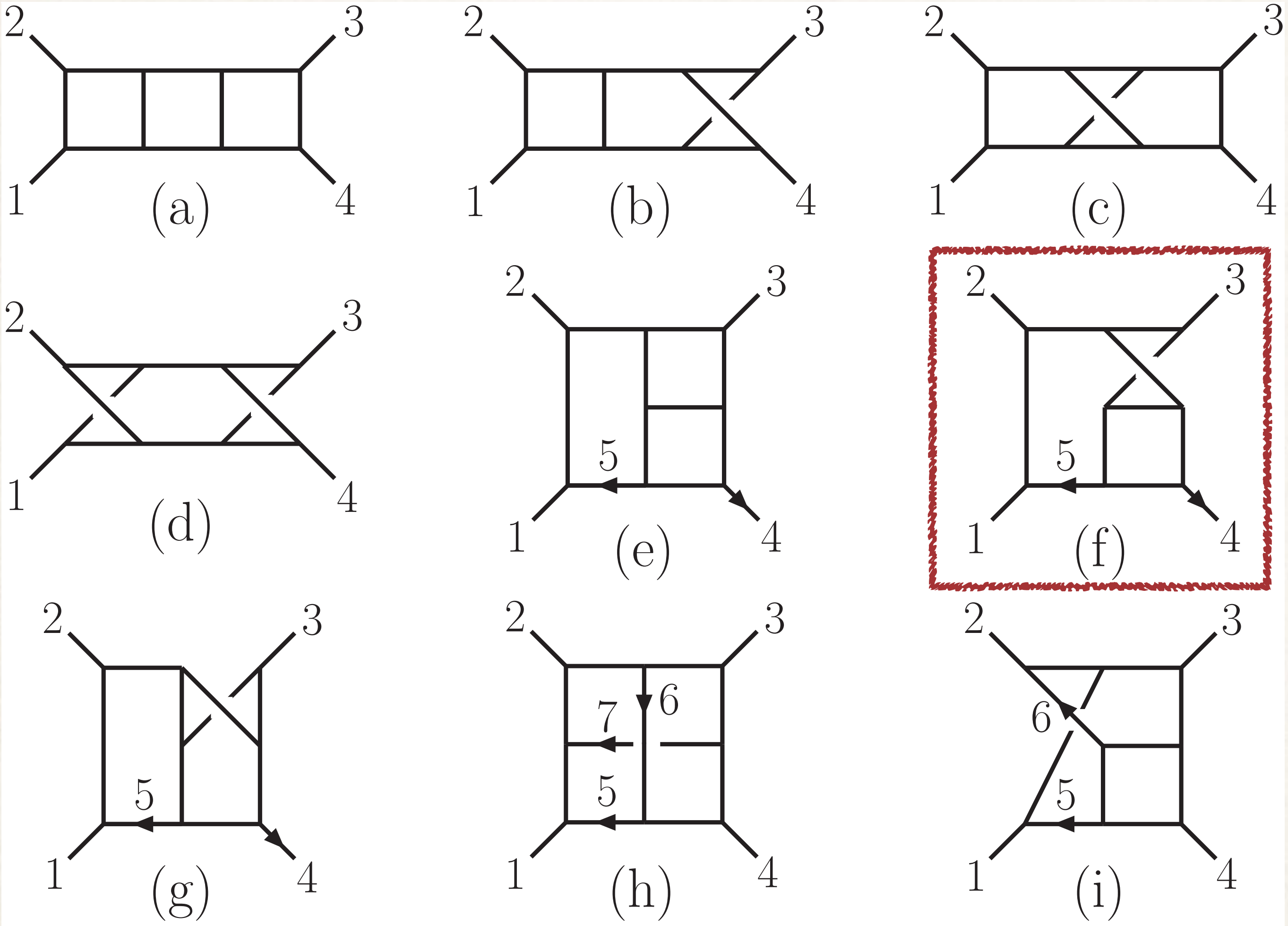


Old numerator

$$N = (\ell_5 + k_4)^2 (k_1 + k_2)^2$$

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude



Old numerator

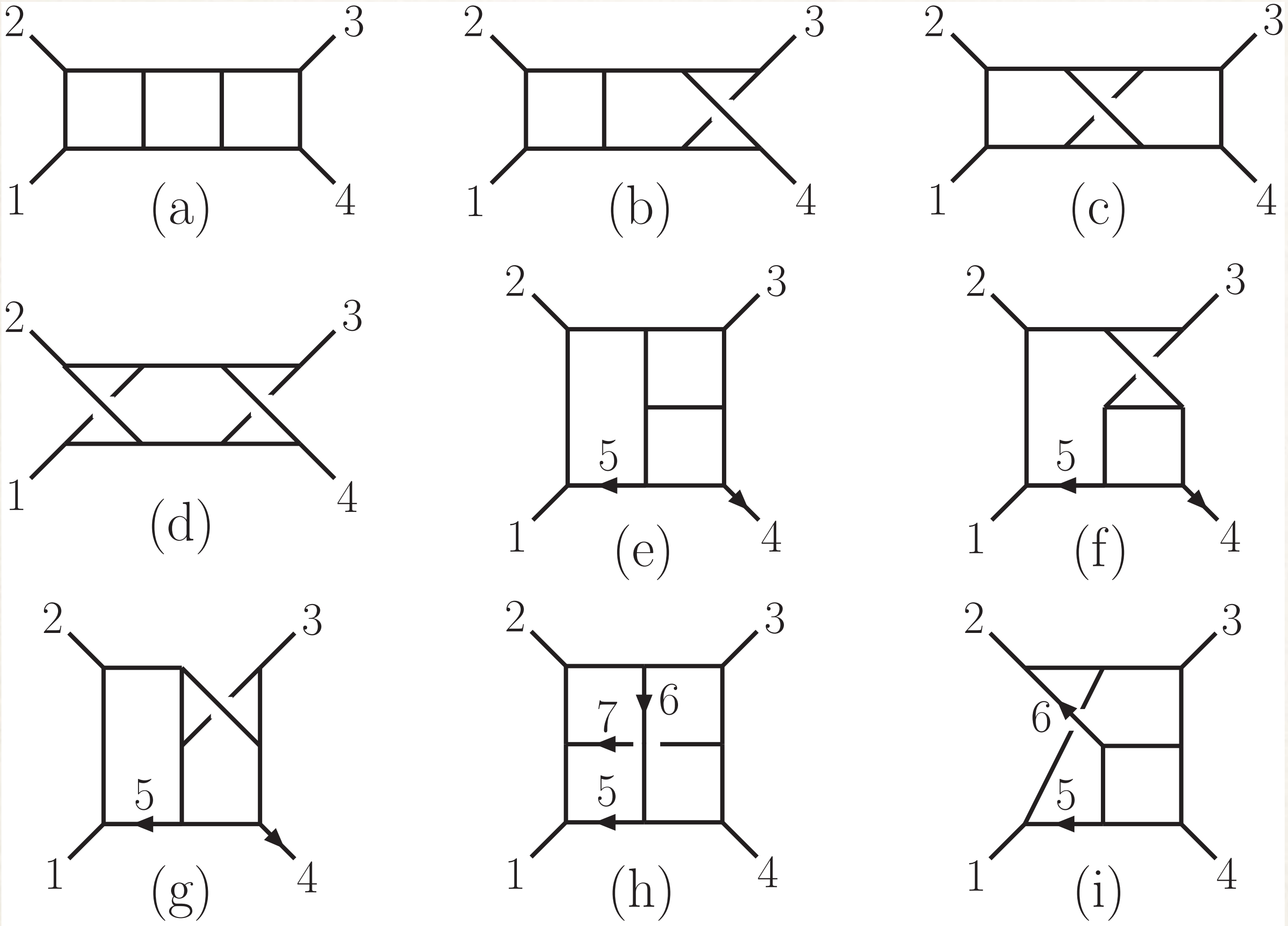
$$N = (\ell_5 + k_4)^2 \boxed{(k_1 + k_2)^2}$$

New numerator

$$N = (\ell_5 + k_4)^2 \boxed{[(\ell_5 + k_3)^2 + (\ell_5 + k_4)^2]}$$

Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude



(Bern, Herrmann, Litsey, Stankowicz, JT 2014)

Numerator	Double pole	Pole at infinity
Original	YES	YES
BCJ	YES	YES
New	NO	NO

Expansion of the amplitude:

YES

Towards scattering inequalities

- Standard approach:

  Non-zero RHS

  $$\mathrm{Cut}(I) = \ldots$$

  - Unitarity cut
  - Maximal cut
  - Leading singularity

- Proposal: $\boxed{\text{Illegal cuts } \mathrm{Cut}(I) = 0 \text{ fix uniquely result!}}$

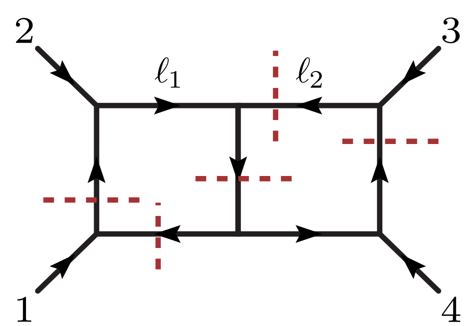
  (up to an overall constant)

  - Underlying inequalities
  - Existence of geometric construction
  - Good variables missing

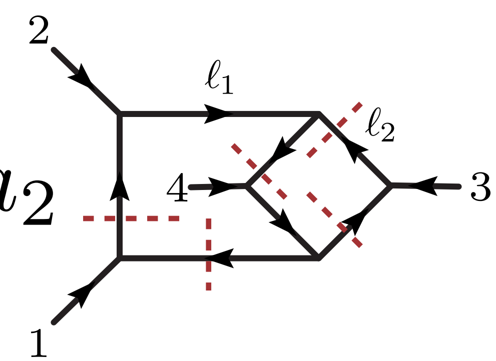
Explicit check

- Two-loop amplitude

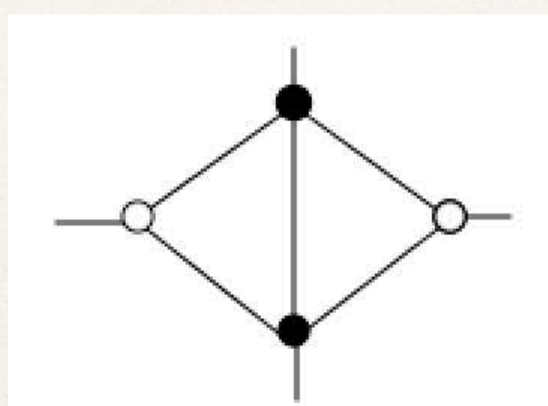
$$M_2 = \sum_{\sigma} \; a_1$$



$$+ \; a_2$$



Illegal 5-cut



$k = 1$

Fixes relative coefficient

$$a_1 = a_2$$

Also three-loop construction

Ratio function

- Integrand: positive value for "positive kinematics"

(Arkani-Hamed, Hodges, Trnka 2014)

- Conjecture: it is true for final amplitudes

- Ratio function: IR finite quantity $R_n = A_n^{(k)}/A_n^{(0)}$

$$R_6 = \frac{1}{2}\left\{ [(1)-(2)+(3)]\, H_1 + [(2)-(3)+(4)]\, H_2 + [(3)-(4)+(5)]\, H_3 \right\}$$

where

$$H_1 = \frac{1}{2}\left[\log u \log v + \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) - 2\zeta_2\right]$$

- Exhaustive numerical check, up to 4-loops

(Dixon, von Hippel, McLeod, Trnka, in progress)

Maximal N=8 supergravity

- UV properties unknown: Is it finite?
- Checked up to 4-loops, problem starts at 7-loops

  (Bern, Carrasco, Dixon, Johansson, Roiban)
- If finite: hidden in the structure of the integrand
  - Natural candidate: poles at infinity — present at 3-loops
  - More detailed study of these poles needed
- Evidence for N<8 theories: enhanced cancelations

  next talks

Thank you for your attention