Singularity structure of maximally supersymmetric scattering amplitudes

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1212.5605, 1312.2007, 1410.0354, 1412.8478
1412.8584, in progress
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Scattering Inequalities

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Plan of the talk

- Integrand of scattering amplitudes
- Planar N=4 SYM: Amplituhedron
- Non-planar extension
- Positivity of ratio function
- N=8 SUGRA and poles at infinity
Scattering Amplitudes

- Basic objects in Quantum Field Theory
- Predictions for colliders: cross-sections
- My motivation: new ideas in QFT
  - Find hidden properties of amplitudes
  - Exploit them in new methods of calculation
  - Generalize to other cases
Integrand

- Finite well-defined rational function before integration

\[ I(\ell_j, k_i, s_i) \quad \text{sum of Feynman diagrams} \]

\[ \Omega = d^4\ell_1 \ldots d^4\ell_L I(\ell_j, k_i, s_i) \quad A = \int_{\ell_j \in \mathbb{R}} \Omega \]

- Qualitative information about the final amplitudes
  - Collinear limits: IR divergencies
  - Poles at infinity: UV structure
  - Types of singularities: transcendental properties
Integrand

- Fixed by principles of locality and unitarity
- Re-express the integrand in the basis of integrals
  \[ I = \sum_j c_j I_j \]
- Fix coefficients using cuts
  \[ \ell^2 = (\ell + Q)^2 = 0 \]
- Maximal cuts, leading singularities:
  (Bern, Dixon, Kosower)
Planar limit

- The integrand defined as a sum of diagrams
  - No global loop momenta
  - Each diagram: its own labels

- Planar limit: dual variables

\[
\begin{align*}
k_1 &= (x_1 - x_2) & k_2 &= (x_2 - x_3) \\
\ell_1 &= (x_3 - y_1) & \ell_2 &= (y_2 - x_3)
\end{align*}
\]

Global labels
Integrand: **single** function
Conditions on the amplitude

**Standard methods**

- Planar diagrams
- Locality + Planarity
- Match physical cuts/singularities

**Alternative**

- Same set of conditions
- Packaged in a different way

\[
\text{Cut}(I) = \begin{array}{c}
1 \\
4 \\
3 \\
2 
\end{array}
\]

Unitarity

Construction not known in general

Complete set known
Toy model: N=4 SYM

- “Simplest Quantum Field Theory”
- Toy model for QCD
  - Tree-level amplitudes identical
  - Loop amplitudes simpler (results up to 7-loops)
- Planar: conformal + dual conformal, convergent series
- Past: new methods for amplitudes originated here
Volume of polyhedron

(Hodges 2009)

- New kinematical variables — momentum twistors
  \[ Z \in \mathbb{C}^3 \]
- Tree-level process: \( gg \rightarrow 5g \)
- Comparison of two calculations of recursion relations
The Amplituhedron

(Arkani-Hamed, JT 2013)

* Generalization of polyhedra to Grassmannian

\[ \mathcal{Y} = C \cdot Z \]

Amplituhedron  Positive matrices: \( \begin{vmatrix} * & * \\ * & * \end{vmatrix} > 0 \)
Minors are positive

* Integrand in planar N=4 SYM: volume of this space

- Geometry labeled by three labels
- Derivation: locality and unitarity
- Check against reference data

\( n \)  number of particles
\( k \)  helicity index
\( \ell \)  number of loops
Inequalities

- **Volume:** logarithmic form  \( \Omega \sim \frac{dx}{x} \) near \( x = 0 \)

- **Amplituhedron variables**  \( z_i \)

\[
(p_i, \epsilon_j, \ell_k) \rightarrow (x_i, \tilde{\eta}_j, y_k) \rightarrow (Z_i, \eta_j, Z_{AB}^{(k)}) \rightarrow z_i
\]

- **Inequalities**  \( P_j(z_i) \geq 0 \)
Legal and illegal boundaries

- Cuts of the amplitude: localize $z_i$

- Inequalities hold $P_j(z_i) \geq 0$; $\ell_k \in \mathbb{C} \iff z_i > 0$
  - Point inside the Amplituhedron space
  - Physical cut or singularity of the amplitude

- One or more inequalities violated $P_j(z_i) < 0$
  - Point outside the Amplituhedron space
  - Unphysical cut or singularity of the amplitude
Example 1: One-loop amplitude

- Consider 4pt one-loop amplitude
- Inequalities: \( z_1, z_2, z_3, z_4 \geq 0 \)
- Boundaries of the space: \( z_1, z_2, z_3, z_4 = (0, \infty) \)
- Differential form
  \[
  \Omega = \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3} \frac{dz_4}{z_4}
  \]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ \begin{array}{c}
\top & \rightarrow & 2 \\
\down & \leftarrow & 3 \\
1 & \rightarrow & 4 \\
\end{array} \]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ z_1 = 0 \]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ z_1 = 0 \]
\[ z_2 = 0 \]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ z_1 = 0 \]
\[ z_2 = 0 \]
\[ z_3 = 0 \]
Example 1: One-loop amplitude

Cuts of the amplitude

\[ z_4 = 0 \]

\[ z_1 = 0 \]
\[ z_2 = 0 \]
\[ z_3 = 0 \]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ z_4 = 0 \]

\[ z_1 = 0 \]
\[ z_2 = 0 \]
\[ z_3 = 0 \]

\[ z_4 = \infty \]

\[ \ell \rightarrow \infty \]

\[ z_4 \in \mathbb{C} \]

“no-triangle”
Example 2: Two-loop amplitude

- Consider 4pt two-loop amplitude

- Inequalities: \( z_1, z_2, z_3, z_4 \geq 0 \)
  \( z_5, z_6, z_7, z_8 \geq 0 \)

\[
(z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0
\]
Example 2: Two-loop amplitude

- Consider 4pt two-loop amplitude

- Inequalities:
  \[ z_1, z_2, z_3, z_4 \geq 0 \]
  \[ z_5, z_6, z_7, z_8 \geq 0 \]
  \[ (z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0 \]

- Check: one-loop cut
  \[ z_1 = 0 \]
  \[ z_2 = 0 \]
  \[ z_3 = 0 \]
  \[ z_4 = 0 \]
  \[ -z_5 z_6 - z_7 z_8 \geq 0 \]

Ω vanishes on this cut
Example 3: Unitarity cut

- **Standard formulation**

  \[ P_j(z_i) \geq 0 \quad i = 1, \ldots, m \]

  \[ z_1 = z_2 = 0 \]

  \[ P_j^{(1)}(z_i) \geq 0 \quad i = 3, \ldots, k \]

  \[ P_j^{(2)}(z_i) \geq 0 \quad i = k + 1, \ldots, m \]

  where \( k \) is a free parameter

- **Set of inequalities split into two sets**

  \[ \text{Cut } M_{n,\ell} = \sum_{\ell_1 + \ell_2 = \ell - 1} M_{n_1,\ell_1} M_{n_2,\ell_2} \]
Physics vs geometry

Standard methods

- Planar diagrams
- Locality + Planarity
- Match physical cuts/singularities

\[ \text{Unitarity} \]

\[ \text{Cut}(I) = \]

Construction not known in general

Amplituhedron

- Inequalities
  \[ P_j(z_i) \geq 0 \]
- Logarithmic form
  \[ \Omega \sim \frac{dx}{x} \]

Complete set known
Non-planar amplitudes

* No global variables: standard $k_i, \ell_k$

* No single form, sum of diagrams

\[ \Omega = \sum_{\sigma, j} C_j \cdot \Omega_j(k_i, \ell_k) \]

$C_j$ color factor

* Each has its own variables
Constraints

- Inspired by the planar sector we conjecture:
  - Logarithmic singularities $\Omega \sim \frac{dx}{x}$
  - No poles at $\ell \to \infty$

- Stronger condition: each diagram individually
  \[
  I_j(k_i, \ell_k) = \frac{N_j(k_i, \ell_k)}{P_1^2 P_2^2 \ldots P_m^2}
  \]

- Find the basis and expand the amplitude
Evidence 1: Two-loop amplitude

Expansion of the 4pt two-loop amplitude

Two basis integrals

\[ N_1 = (k_1 + k_2)^2 \]

\[ N_2 = (k_1 + k_2)^2 \]

Double Poles

Poles at infinity
Evidence 1: Two-loop amplitude

Expansion of the 4pt two-loop amplitude

(Bern, Rozowsky, Yan 1997)

Two basis integrals

\[ N_1 = (k_1 + k_2)^2 \]

\[ N_2 = (k_1 + k_2)^2 \]

Double Poles

NO

YES

Poles at infinity

NO

YES
Evidence 1: Two-loop amplitude

\[
I = \frac{\frac{d^4 \ell_1 d^4 \ell_2 (p_1 + p_2)^2}{\ell_1^2 (\ell_1 - k_2)^2 (\ell_1 - k_1 - k_2)^2 \ell_2^2 (\ell_2 - k_3)^2 (\ell_1 + \ell_2)^2 (\ell_1 + \ell_2 + k_4)^2}}}{\ell_1^2 (\ell_1 - k_2)^2 (\ell_1 - k_1 - k_2)^2 \ell_2^2 (\ell_2 - k_3)^2 (\ell_1 + \ell_2)^2 (\ell_1 + \ell_2 + k_4)^2}
\]

Perform cuts

\[
\ell_2^2 = (\ell_2 - k_3)^2 = (\ell_1 + \ell_2)^2 = (\ell_1 + \ell_2 + k_4)^2 = 0
\]

Localize \( \ell_2 \) completely
Evidence 1: Two-loop amplitude

\[
\text{Cut}_1 dI = \frac{d^4 \ell_1}{\ell_1^2 (\ell_1 - k_2)^2 (\ell_1 - k_1 - k_2)^2 [(\ell_1 + k_3)^2 (\ell_1 + k_4)^2 - \ell_1^2 (\ell_1 + k_3 + k_4)^2]}
\]

Localize \( \ell_1 = \alpha k_2 \) by cutting \( \ell_1^2 = (\ell_1 - k_2)^2 = 0 \) and the Jacobian
Evidence 1: Two-loop amplitude

\[ \text{Cut}_{1,2} \, dI = \frac{d\alpha}{(\alpha + 1)\alpha^2 tu} \]

- Double pole for \( \alpha = 0 \)
- There is also pole at infinity
- We want to find a numerator which cancels all that
Evidence 1: Two-loop amplitude

\[
\text{Cut}_{1,2} \, dI = \frac{d\alpha}{(\alpha + 1)\alpha^2 tu}
\]

New numerator

\[
N = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2
\]

Double pole for \( \alpha = 0 \)

Cancels double pole

\( N \rightarrow \alpha s \)
Evidence 1: Two-loop amplitude

- **New expansion of the 4pt two-loop amplitude**

  (Arkani-Hamed, Bourjaily, Cachazo, JT, 2014)

  Two basis integrals

\[
N_1 = (k_1 + k_2)^2 \\
N_2 = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2
\]

Double Poles  NO  NO
Poles at infinity  NO  NO

Expand amplitude in the basis:  YES
Evidence 2: Three-loop amplitude

Basis for three-loop four point amplitude

(Bern, Carrasco, Dixon, Johansson, Kosower 2007)

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Double pole</th>
<th>Pole at infinity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>BCJ</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>
Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

Old numerator

\[ N = (\ell_5 + k_4)^2 (k_1 + k_2)^2 \]
Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

\[
N = (\ell_5 + k_4)^2(k_1 + k_2)^2
\]

Old numerator

\[
N = (\ell_5 + k_4)^2[(\ell_5 + k_3)^2 + (\ell_5 + k_4)^2]
\]

New numerator
Evidence 2: Three-loop amplitude

Basis for three-loop four point amplitude

(Bern, Herrmann, Litsey, Stankowicz, JT 2014)

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Expansion of the amplitude: YES
Towards scattering inequalities

- **Standard approach:**
  Non-zero RHS
  \[ \text{Cut}(I) = \ldots \]
  \* Unitarity cut
  \* Maximal cut
  \* Leading singularity

- **Proposal:** Illegal cuts \[ \text{Cut}(I) = 0 \] fix uniquely result!
  (up to an overall constant)
  \* Underlying inequalities
  \* Existence of geometric construction
  \* Good variables missing
Explicit check

- Two-loop amplitude

\[ M_2 = \sum_{\sigma} a_1 \]

Illegal 5-cut

Fixes relative coefficient

\[ a_1 = a_2 \]

Also three-loop construction

\[ k = 1 \]
Ratio function

- Integrand: positive value for "positive kinematics"
  (Arkani-Hamed, Hodges, Trnka 2014)
- Conjecture: it is true for final amplitudes
- Ratio function: IR finite quantity
  \[ R_n = A_n^{(k)} / A_n^{(0)} \]
  \[ R_6 = \frac{1}{2} \left\{ [(1) - (2) + (3)] H_1 + [(2) - (3) + (4)] H_2 + [(3) - (4) + (5)] H_3 \right\} \]
  where
  \[ H_1 = \frac{1}{2} [\log u \log v + \text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w) - 2\zeta_2] \]
- Exhaustive numerical check, up to 4-loops
  (Dixon, von Hippel, McLeod, Trnka, in progress)
Maximal $N=8$ supergravity

- UV properties unknown: Is it finite?
- Checked up to 4-loops, problem starts at 7-loops
  
  (Bern, Carrasco, Dixon, Johansson, Roiban)

- If finite: hidden in the structure of the integrand
  
  - Natural candidate: poles at infinity — present at 3-loops
  - More detailed study of these poles needed

- Evidence for $N<8$ theories: enhanced cancelations

next talks
Thank you for your attention