

Relation between the QCD resummation procedures for transverse-momentum distributions

— " b_* " vs. "complex b " —

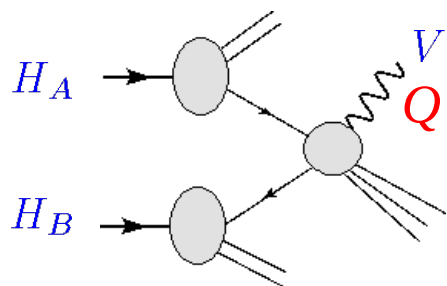
Kazuhiro Tanaka (Juntendo U)

Hadroproduction of vector boson

Drell, Yan ('70)

$$x_{A,B} = \frac{Q}{\sqrt{S}} e^{\pm y}$$

$$H_A + H_B \rightarrow V(Q, Q_T, y, \dots) + X, \quad V = Z, W, \gamma^*$$

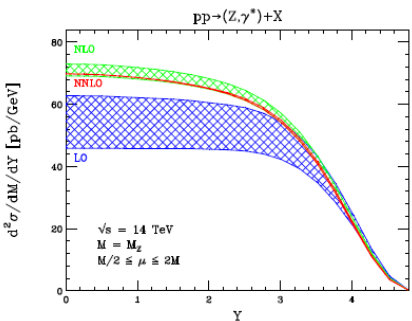


- at hadron colliders
- constraints for PDFs
- new physics search

$$d\sigma \propto \sum_{j=q,\bar{q}} e_j^2 f_{j/A}(x_A, Q^2) f_{\bar{j}/B}(x_B, Q^2) + \dots$$

Perturbative QCD corrections (up to NNLO):

- Total cross section
Hamberg, Matsuura, van Neerven ('91), Harlander, Kilgore ('02)
- Rapidity distribution
Anastasiou, Dixon, Melnikov, Kilgore ('02)
- Fully differential cross section
Melnikov, petriello ('02), Catani, Cieri, Ferrera, de Florian, Grazzini ('09)



Anastasiou et al.

NLO EW corrections

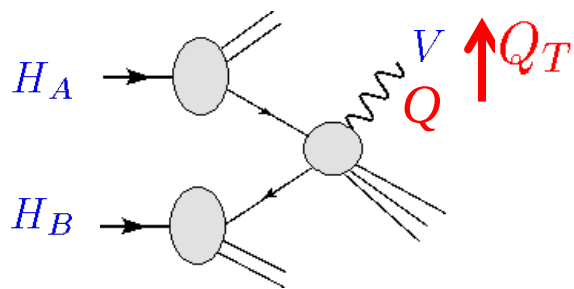
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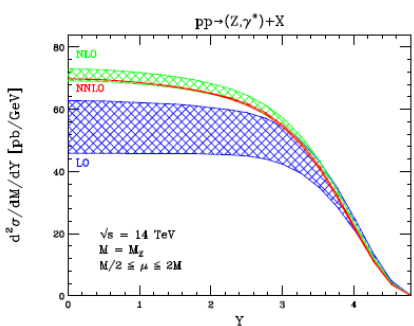


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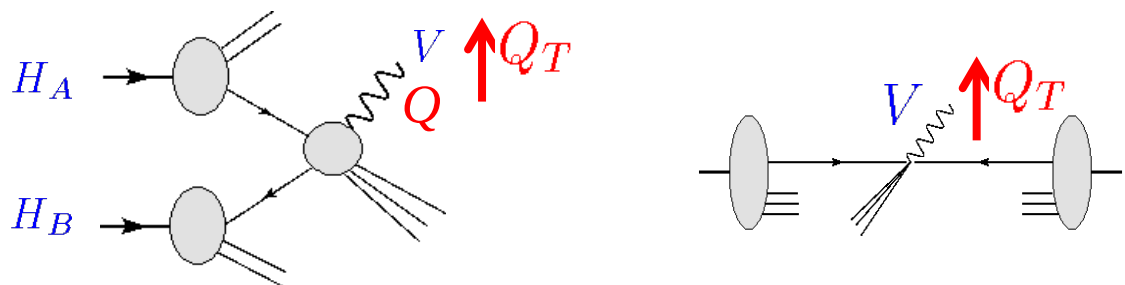
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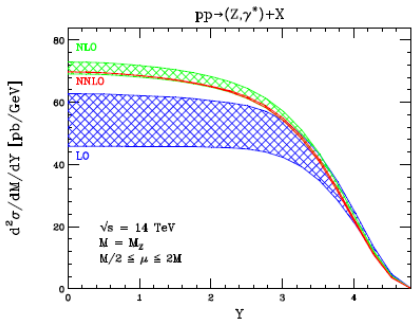


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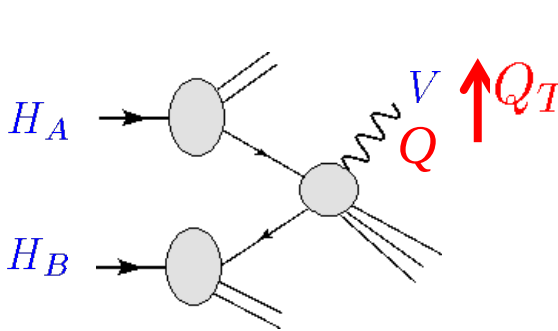
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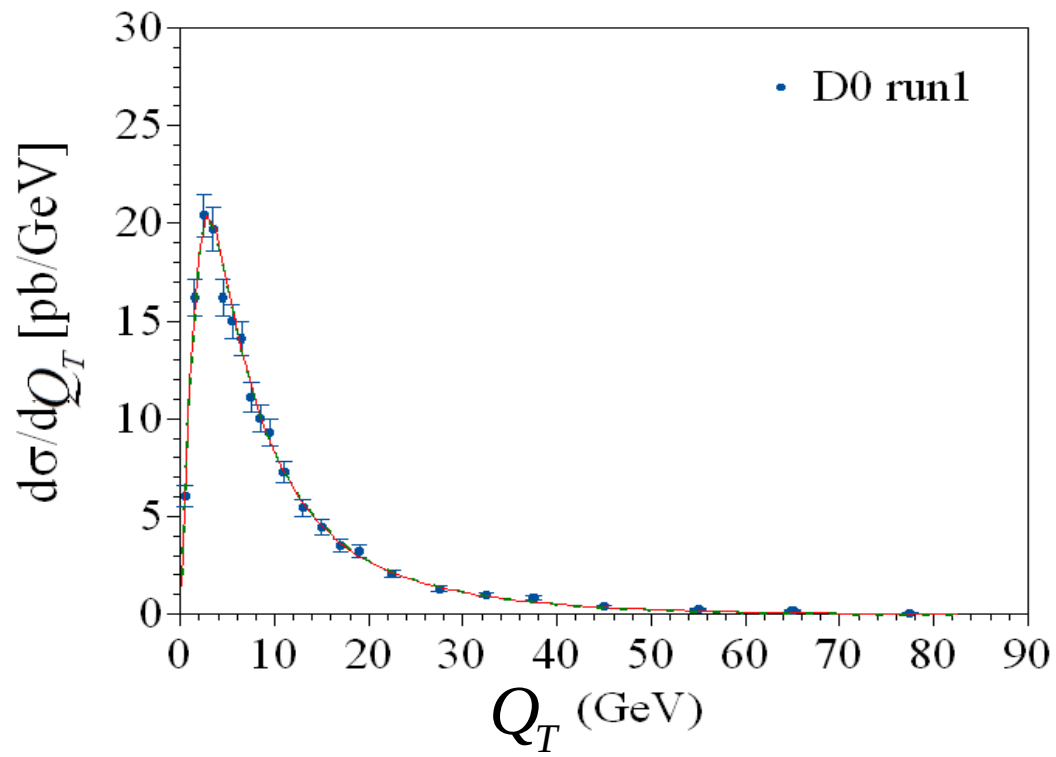
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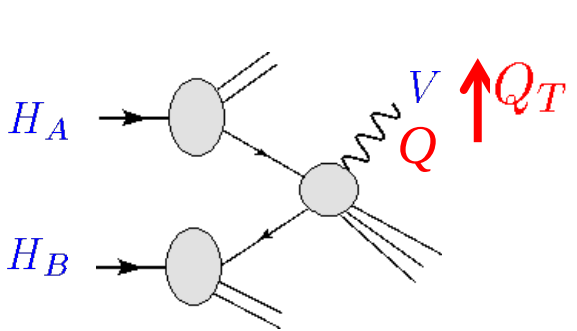


$$V = Z, W, \gamma^*$$

mostly produced at small Q_T

peak at 4-6 GeV for Z production

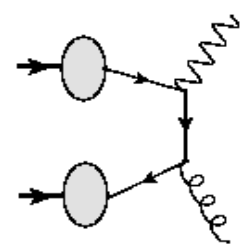
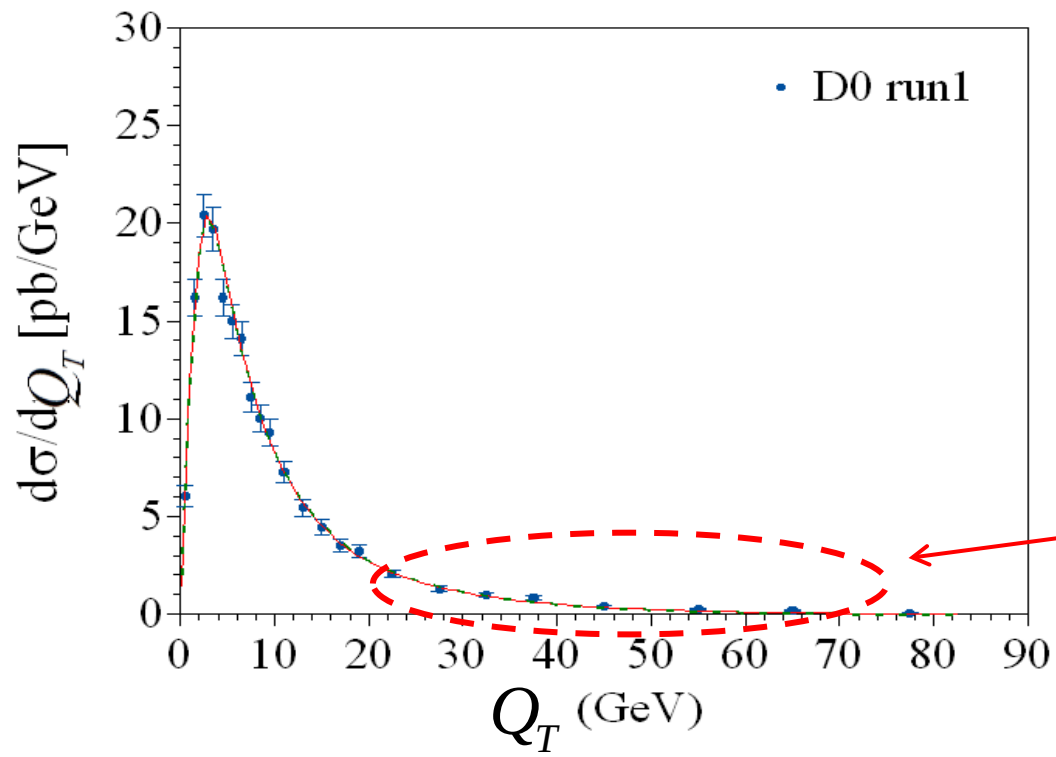




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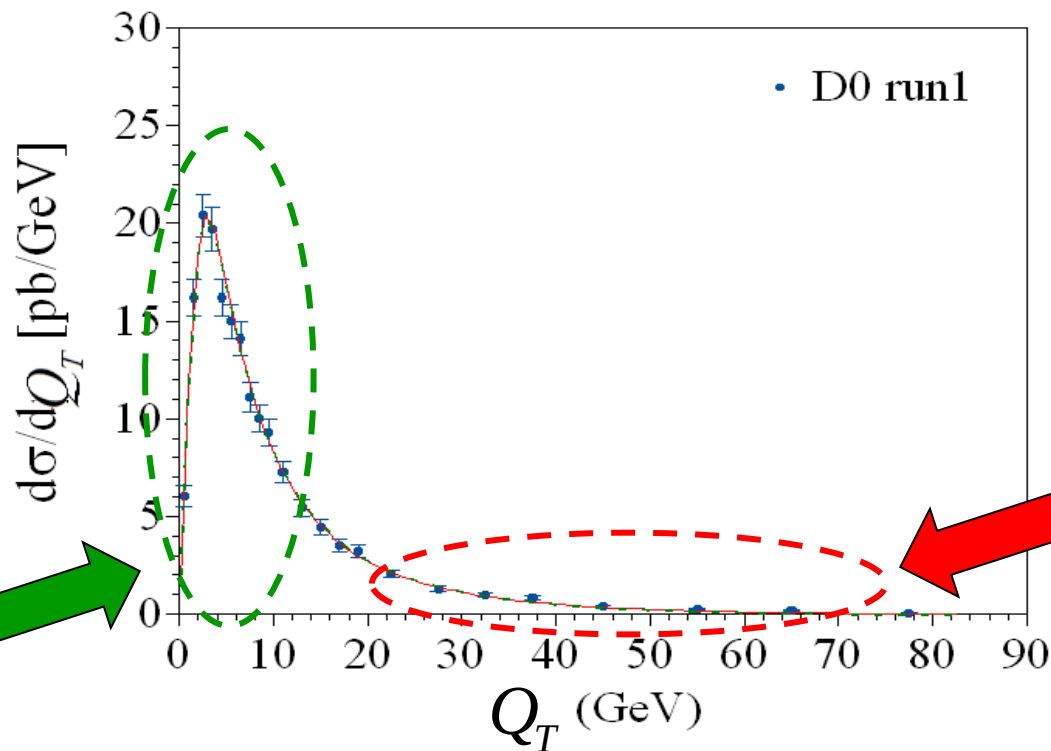


hard
gluon
emission

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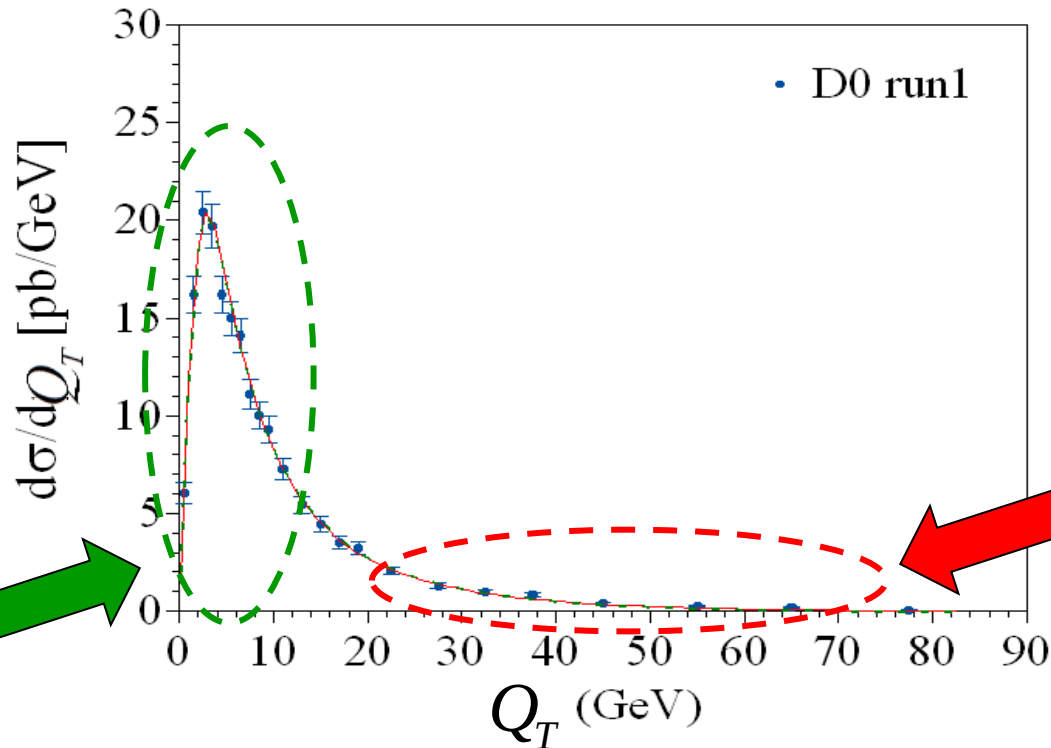
soft gluon emission

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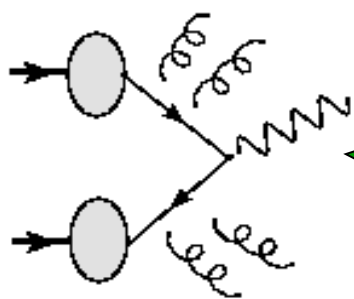
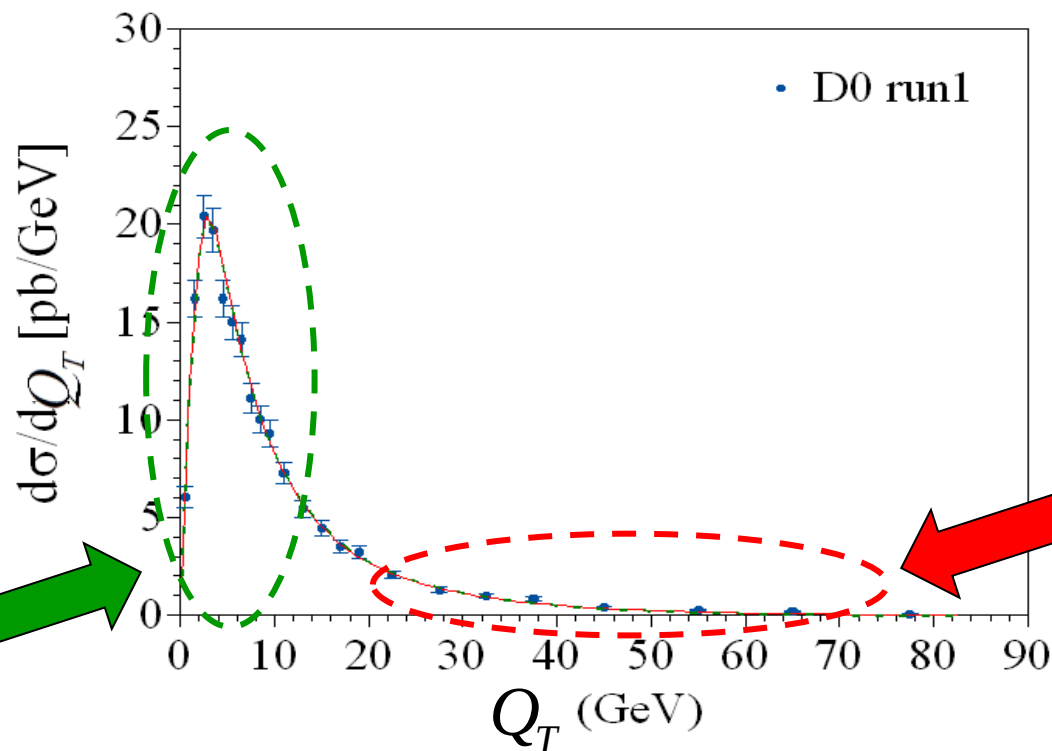
hard
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$$\alpha_s \ln \frac{Q}{Q_T}, \quad \alpha_s \ln^2 \frac{Q}{Q_T}$$

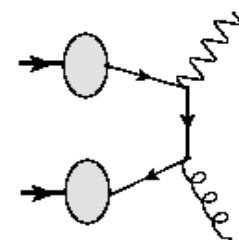
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soft gluon emission
resummation



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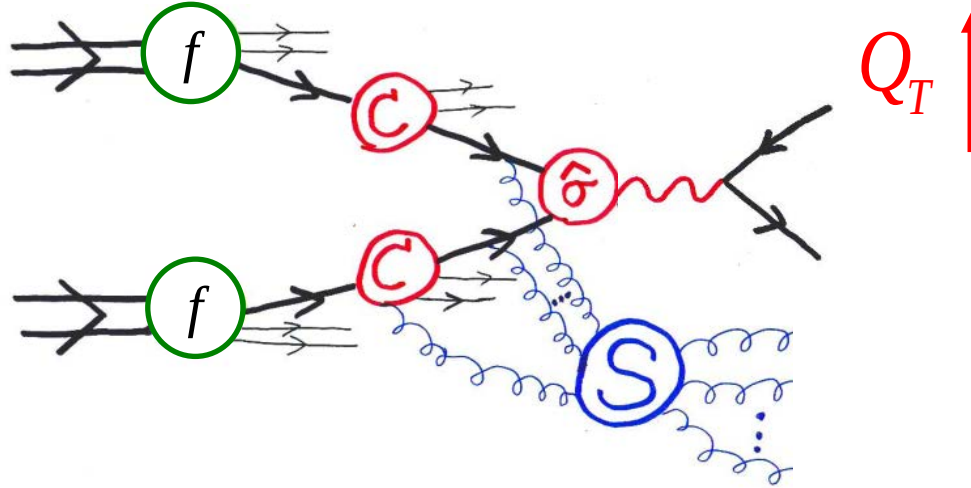
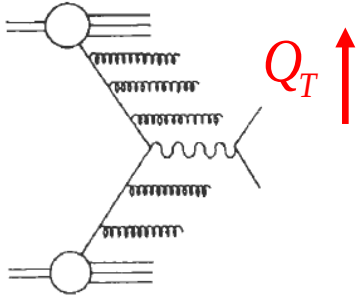
fixed-order
perturbation theory

all-orders resummation

Altarelli, Ellis, Greco, Maltinelli ('84)

Collins, Soper, Sterman ('85)

Catani, de Florian, Grazzini ('01)

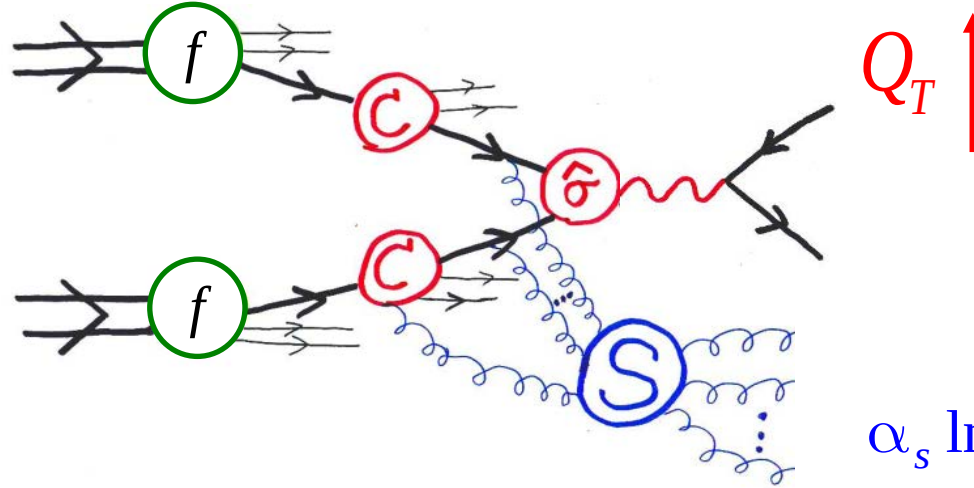
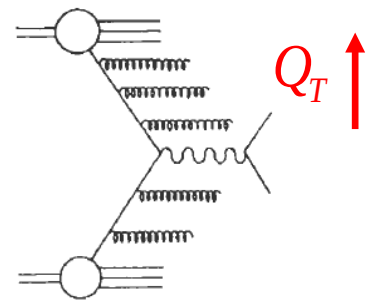


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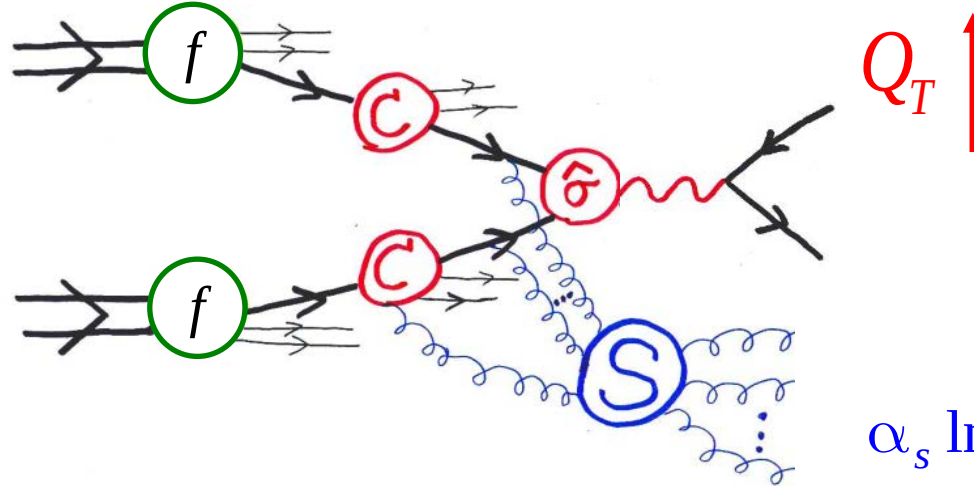
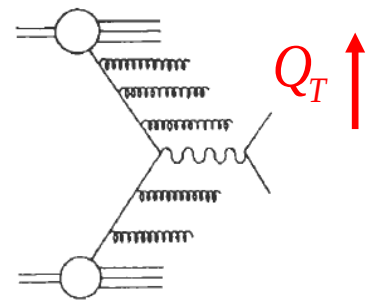
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all-orders resummation



$$\alpha_s \ln^2 \frac{Q}{Q_T}, \quad \alpha_s \ln \frac{Q}{Q_T}$$

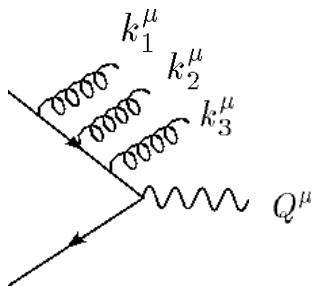
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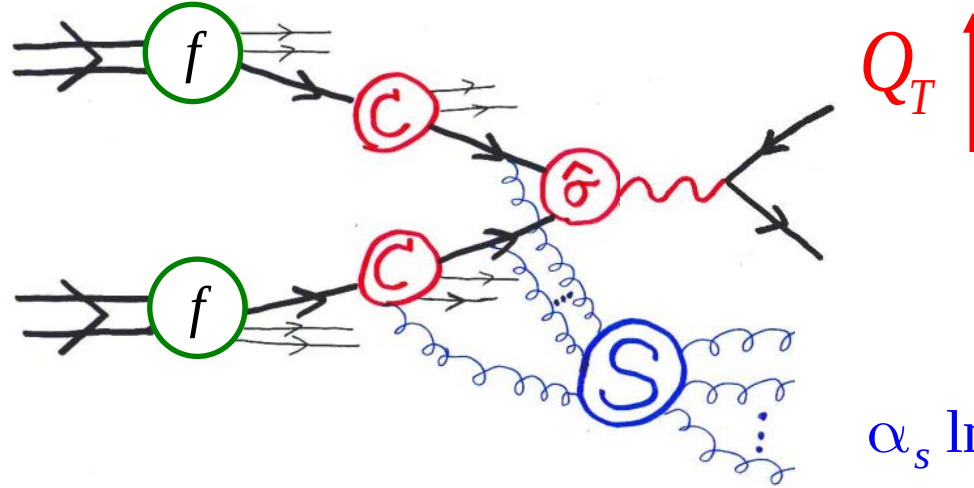
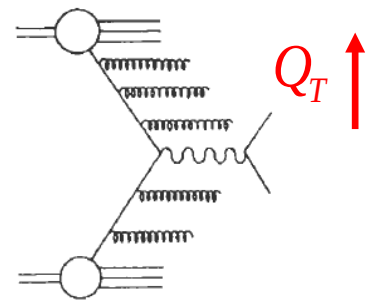
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Impact parameter b space

$$\delta^{(2)}(Q_T - k_{1T} - k_{2T} - \dots - k_{nT}) \rightarrow \int d^2 b e^{ibQ_T} \prod_n e^{-ib \cdot k_T}$$



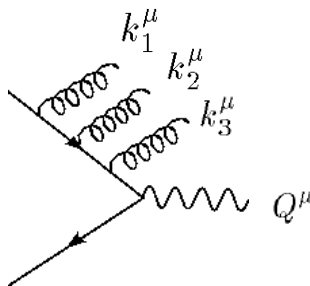
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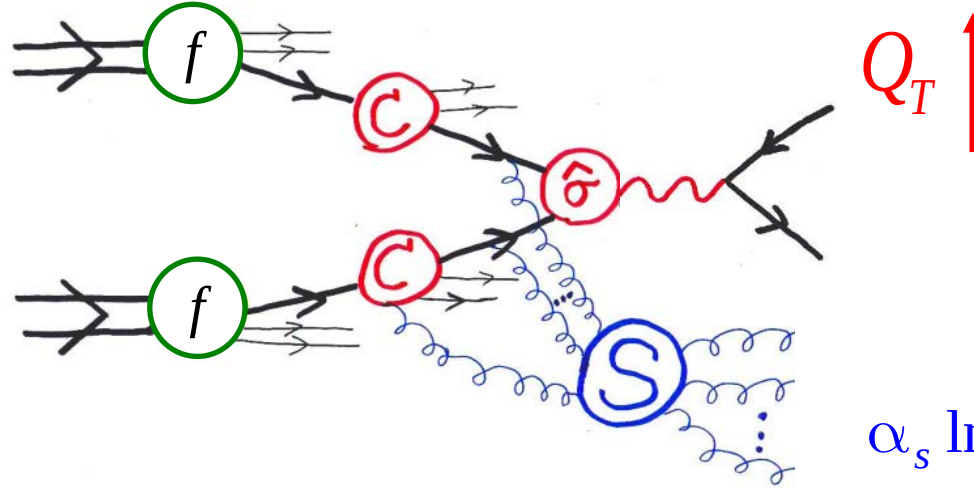
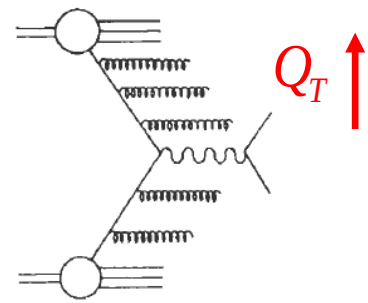
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$$\alpha_s \ln \frac{Q^2 b^2}{b_0^2} \equiv \alpha_s L$$

$$b_0 = 2e^{-\gamma_E}$$

all-orders resummation



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$$d\sigma \propto \sum_{j=q,\bar{q}} e_j^2 f_{j/A}(x_A, Q^2) f_{\bar{j}/B}(x_B, Q^2) + \dots$$

$$\Rightarrow \int d^2b e^{i\mathbf{b} \cdot \mathbf{Q}_T} \sum_{j,i,k} \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} (C_{ji} \otimes f_{i/A}) \left(x_A, \frac{b_0^2}{b^2} \right) (C_{\bar{j}k} \otimes f_{k/B}) \left(x_B, \frac{b_0^2}{b^2} \right) + \dots$$

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$$S_j(b, Q) = - \int_{b_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}$$

$$A_q(\alpha_s) = C_F \frac{\alpha_s}{\pi} + \frac{1}{2} C_F \left\{ \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_G - \frac{5}{9} N_f \right\} \left(\frac{\alpha_s}{\pi} \right)^2 + \dots$$

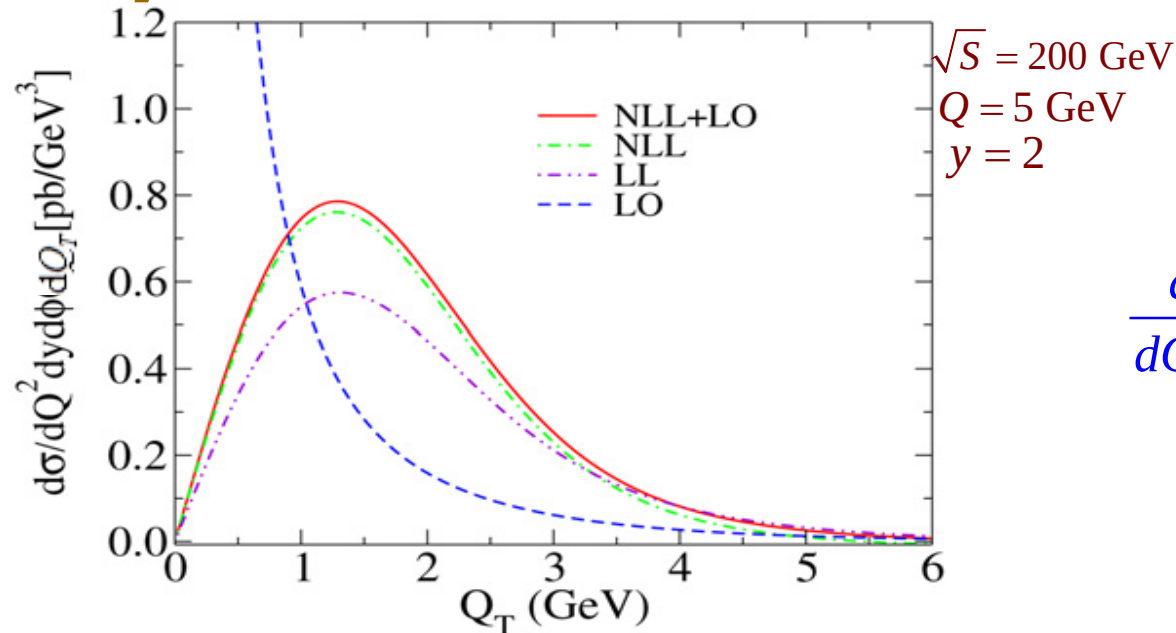
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fully differential Drell-Yan



H. Kawamura, J. Kodaira, KT,
NPB777 ('07) 203

$$\frac{d\sigma^{\text{fixed-order}}}{dQ^2 dy d\phi dQ_T^2} \sim \sum_n \alpha_s^n \left[c_n \delta(Q_T^2) + \sum_{k=0}^{2n-1} d_{nk} \frac{\ln^k(Q^2/Q_T^2)}{Q_T^2} \right]$$

General Structure of Large Logs

LO	1			
NLO	$\alpha_s \mathbf{L}^2$	$\alpha_s \mathbf{L}$	α_s	
NNLO	$\alpha_s^2 \mathbf{L}^4$	$\alpha_s^2 \mathbf{L}^3$	$\alpha_s^2 \mathbf{L}^2$	$+\dots$
N³LO	$\alpha_s^3 \mathbf{L}^6$	$\alpha_s^3 \mathbf{L}^5$	$\alpha_s^3 \mathbf{L}^4$	$+\dots$
N^kLO	$\alpha_s^k \mathbf{L}^{2k}$	$\alpha_s^k \mathbf{L}^{2k-1}$	$\alpha_s^k \mathbf{L}^{2k-2}$	$+\dots$
	LL	NLL	NNLL	

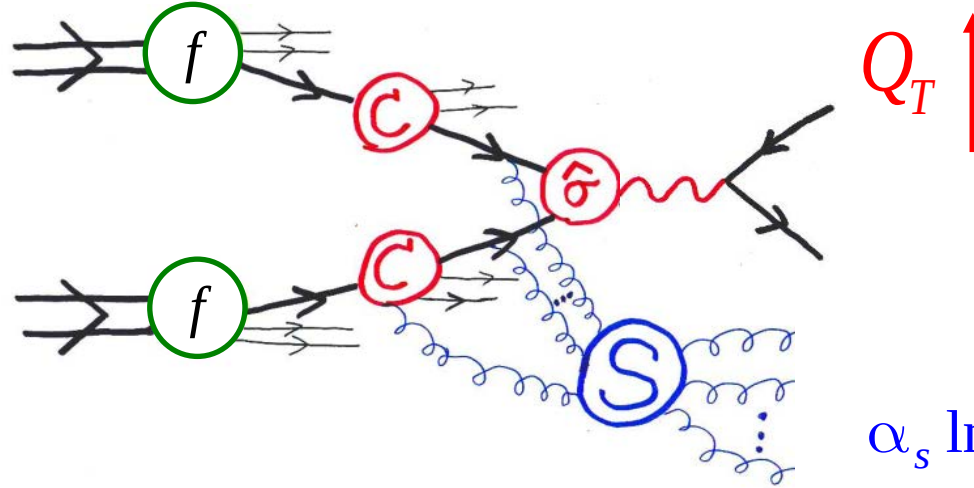
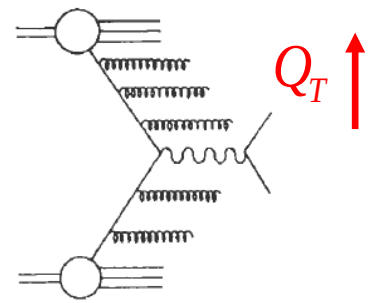
$$\alpha_s \ln \frac{Q^2 b^2}{b_0^2} \equiv \alpha_s L$$

$$\frac{d\sigma}{dQ^2 dQ_T^2 dy} = \frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} + \frac{d\sigma^{(\text{fin})}}{dQ^2 dQ_T^2 dy}$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} W(b; Q, x_A, x_B) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) W(b; Q, x_A, x_B)$$

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all-orders resummation



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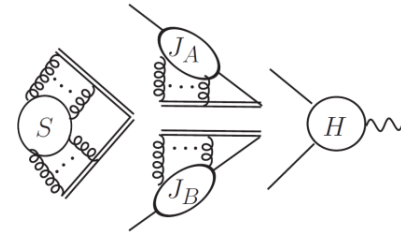
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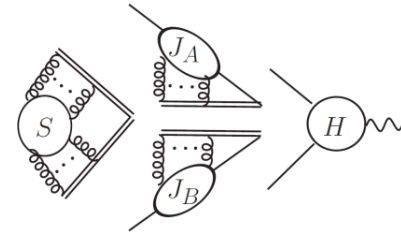


$$\frac{\partial}{\partial \ln Q^2} W_j(b; Q, x_A, x_B) = - \left[\int_{b_0^2/b^2}^{Q^2} \frac{d\kappa^2}{\kappa^2} A_j(\alpha_s(\kappa^2)) + B_j(\alpha_s(Q^2)) \right] W_j(b; Q, x_A, x_B)$$

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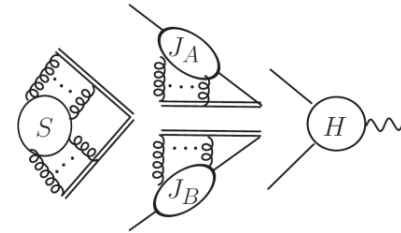
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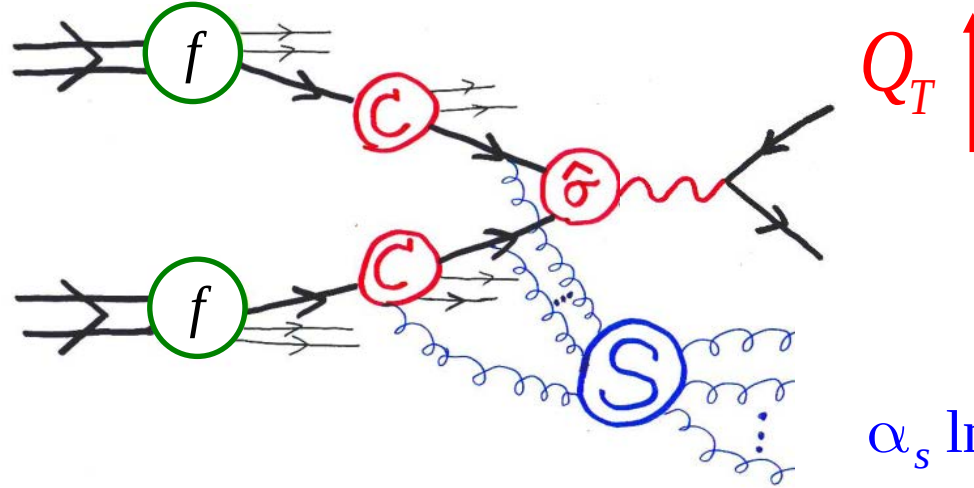
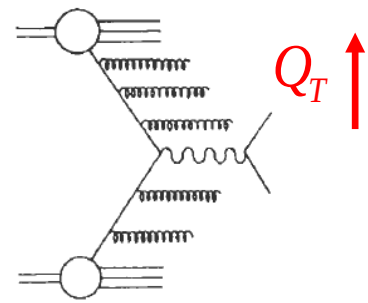
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$$d\sigma \propto \sum_{j=q,\bar{q}} e_j^2 f_{j/A}(x_A, Q^2) f_{\bar{j}/B}(x_B, Q^2) + \dots$$

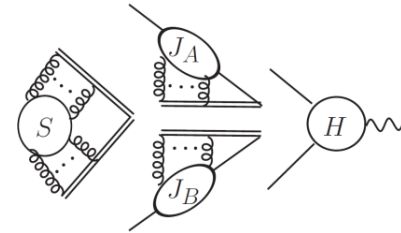
$$\Rightarrow \int d^2b e^{i\mathbf{b} \cdot \mathbf{Q}_T} \sum_{j,i,k} \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} (C_{ji} \otimes f_{i/A}) \left(x_A, \frac{b_0^2}{b^2} \right) (C_{\bar{j}k} \otimes f_{k/B}) \left(x_B, \frac{b_0^2}{b^2} \right) + \dots$$

$$(C_{ji} \otimes f_{i/A})(x, \mu^2) = \int_x^1 \frac{dz}{z} C_{ji}(z, \alpha_s(\mu^2)) f_{i/A}(x/z, \mu^2)$$

$$\frac{d\sigma}{dQ^2 dQ_T^2 dy} = \frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} + \frac{d\sigma^{(\text{fin})}}{dQ^2 dQ_T^2 dy}$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} W(b; Q, x_A, x_B) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) W(b; Q, x_A, x_B)$$

$$W(b; Q, x_A, x_B) = \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) W_j(b; Q, x_A, x_B)$$



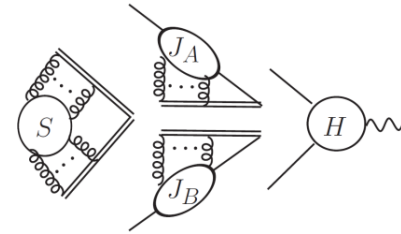
$$\frac{\partial}{\partial \ln Q^2} W_j(b; Q, x_A, x_B) = - \left[\int_{b_0^2/b^2}^{Q^2} \frac{d\kappa^2}{\kappa^2} A_j(\alpha_s(\kappa^2)) + B_j(\alpha_s(Q^2)) \right] W_j(b; Q, x_A, x_B)$$

$$W_j(b; Q, x_A, x_B) = e^{S_j(b, Q)} W_j(b; \frac{b_0}{b}, x_A, x_B)$$

$$\frac{d\sigma}{dQ^2 dQ_T^2 dy} = \frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} + \frac{d\sigma^{(\text{fin})}}{dQ^2 dQ_T^2 dy}$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} W(b; Q, x_A, x_B) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) W(b; Q, x_A, x_B)$$

$$W(b; Q, x_A, x_B) = \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) W_j(b; Q, x_A, x_B)$$



$$\frac{\partial}{\partial \ln Q^2} W_j(b; Q, x_A, x_B) = - \left[\int_{b_0^2/b^2}^{Q^2} \frac{d\kappa^2}{\kappa^2} A_j(\alpha_s(\kappa^2)) + B_j(\alpha_s(Q^2)) \right] W_j(b; Q, x_A, x_B)$$

$$W_j(b; Q, x_A, x_B) = e^{S_j(b, Q)} W_j(b; \frac{b_0}{b}, x_A, x_B)$$

$$b \ll 1/\Lambda_{\text{QCD}} \quad (Q_T \gg \Lambda_{\text{QCD}})$$

$$W_j(b; \frac{b_0}{b}, x_A, x_B) \sim (\mathbf{C}_{ji} \otimes f_{i/A}) \left(x_A, \frac{b_0^2}{b^2} \right) (\mathbf{C}_{\bar{j}k} \otimes f_{k/B}) \left(x_B, \frac{b_0^2}{b^2} \right) \equiv W_j^{(\text{OPE})} (b; \frac{b_0}{b}, x_A, x_B)$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b \, e^{i\mathbf{b}\cdot\mathbf{Q}_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$$W_j(b; \frac{b_0}{b}) \quad \overset{b \ll \frac{1}{\Lambda_{\text{QCD}}}}{\longrightarrow} \quad W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b \, e^{i\mathbf{b} \cdot \mathbf{Q}_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$$W_j(b; \frac{b_0}{b}) \xrightarrow{b \ll \frac{1}{\Lambda_{\text{QCD}}}} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$$

$$\equiv \bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0) \quad \text{TMD}$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$$W_j(b; \frac{b_0}{b}) \xrightarrow{b \ll \frac{1}{\Lambda_{\text{QCD}}}} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$$

$$\equiv \bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0) \quad \text{TMD}$$

$$W_j(b; \frac{b_0}{b}) = W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \frac{W_j(b; \frac{b_0}{b})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \equiv W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{\text{NP}}(b)$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$$W_j(b; \frac{b_0}{b}) \xrightarrow{b \ll \frac{1}{\Lambda_{\text{QCD}}}} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$$

$$\equiv \bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0) \quad \text{TMD}$$

“primordial k_T ”

$$W_j(b; \frac{b_0}{b}) = W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \frac{W_j(b; \frac{b_0}{b})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \equiv W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b)$$

$$F^{NP}(b) = e^{-g_{\text{NP}} b^2}$$

$$S_j(b, Q) = - \int_{b_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}$$

$$A_q(\alpha_s) = C_F \frac{\alpha_s}{\pi} + \frac{1}{2} C_F \left\{ \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_G - \frac{5}{9} N_f \right\} \left(\frac{\alpha_s}{\pi} \right)^2 + \dots$$

$$B_q(\alpha_s) = -\frac{3}{2} C_F \frac{\alpha_s}{\pi} + \dots$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b \, e^{i\mathbf{b} \cdot \mathbf{Q}_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$W_j(b; \frac{b_0}{b})$
 $\equiv$
 $\bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0)$

$b \ll \frac{1}{\Lambda_{\text{QCD}}}$

$W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$

TMD

$$W_j(b; \frac{b_0}{b}) = W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \frac{W_j(b; \frac{b_0}{b})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \equiv W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b)$$

“primordial k_T ”

$F^{NP}(b) = e^{-g_{\text{NP}} b^2}$
 $g_{\text{NP}} = g_1 + g_2 \ln(Q/2Q_0)$

$W_j(b; \frac{b_0}{b})$

$$S_j(b, Q) = - \int_{b_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}$$

$$A_q(\alpha_s) = C_F \frac{\alpha_s}{\pi} + \frac{1}{2} C_F \left\{ \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_G - \frac{5}{9} N_f \right\} \left(\frac{\alpha_s}{\pi} \right)^2 + \dots$$

$$B_q(\alpha_s) = -\frac{3}{2} C_F \frac{\alpha_s}{\pi} + \dots$$

$$= \frac{1}{\alpha_s(\mu_R^2)} h^{(0)}(\lambda) + h^{(1)}(\lambda) + \sum_{n=2}^{\infty} \left(\frac{\alpha_s(\mu_R^2)}{2\pi} \right)^{n-1} h^{(n)}(\lambda)$$

$$h^{(0)}(\lambda) = \frac{A^{(1)}}{2\pi\beta_0^2} [\lambda + \ln(1-\lambda)]$$

$$\lambda = \beta_0 \alpha_s(\mu_R^2) \ln \frac{Q^2 b^2}{b_0^2} \equiv \beta_0 \alpha_s(\mu_R^2) L$$

$$h^{(1)}(\lambda) = \frac{A^{(1)}\beta_1}{2\pi\beta_0^3} \left[\frac{1}{2} \ln^2(1-\lambda) + \frac{\lambda + \ln(1-\lambda)}{1-\lambda} \right] + \frac{B^{(1)}}{2\pi\beta_0} \ln(1-\lambda) - \frac{1}{4\pi^2\beta_0^2} \left[A^{(2)} - 2\pi\beta_0 A^{(1)} \ln \frac{Q^2}{\mu_R^2} \right] \left[\frac{\lambda}{1-\lambda} + \ln(1-\lambda) \right]$$

$$\text{Landau pole} \quad b_{LP} \sim \frac{1}{Q} e^{\frac{1}{2\beta_0\alpha_s(Q^2)}}$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b \, e^{i\mathbf{b} \cdot \mathbf{Q}_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$W_j(b; \frac{b_0}{b})$
 $\equiv$
 $\bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0)$

$\xrightarrow{b \ll \frac{1}{\Lambda_{\text{QCD}}}}$

$W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$

TMD

$$W_j(b; \frac{b_0}{b}) = W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \frac{W_j(b; \frac{b_0}{b})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \equiv W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b)$$

“primordial k_T ”

$F^{NP}(b) = e^{-g_{NP} b^2}$

$g_{NP} = g_1 + g_2 \ln(Q/2Q_0)$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$$W_j(b; \frac{b_0}{b}) \xrightarrow{b \ll \frac{1}{\Lambda_{\text{QCD}}}} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$$

$$= \bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0) \quad \text{TMD}$$

$$W_j(b; \frac{b_0}{b}) = W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \frac{W_j(b; \frac{b_0}{b})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \equiv W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{\text{NP}}(b)$$

“primordial k_T ”

$$F^{\text{NP}}(b) = e^{-g_{\text{NP}} b^2}$$

$$g_{\text{NP}} = g_1 + g_2 \ln(Q/2Q_0)$$

$$1. e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \rightarrow e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) \quad b_* = \frac{b}{\sqrt{1 + b^2/b_{\text{max}}^2}}$$

J. Collins, D. Soper,
G. Sterman ('85)

A. Kulesza, W. Stirling ('02)

C. Balazs, C. Yuan ('00)

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$$W_j(b; \frac{b_0}{b}) \xrightarrow{b \ll \frac{1}{\Lambda_{\text{QCD}}}} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$$

$$= \bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0) \quad \text{TMD}$$

$$W_j(b; \frac{b_0}{b}) = W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \frac{W_j(b; \frac{b_0}{b})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \equiv W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{\text{NP}}(b)$$

“primordial k_T ”

$$F^{\text{NP}}(b) = e^{-g_{\text{NP}} b^2}$$

$$g_{\text{NP}} = g_1 + g_2 \ln(Q/2Q_0)$$

1. $e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \rightarrow e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) \quad b_* = \frac{b}{\sqrt{1+b^2/b_{\text{max}}^2}}$

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2. $\int_0^\infty db \rightarrow \int_{C_+ + C_-} db$

Landau pole

$\frac{1}{Q} e^{\frac{1}{2\beta_0\alpha_s(Q^2)}}$

$J_0(bQ_T) = \frac{H_0^{(1)}(bQ_T) + H_0^{(2)}(bQ_T)}{2}$

E. Laenen, G. Sterman, W. Vogelsang ('00)
A. Kulesza, G. Sterman, W. Vogelsang ('02)
G. Bozzi, S. Catani, D. de Florian, M. Grazzini ('02)
H. Kawamura, J. Kodaira, KT ('07)

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) F^{NP}(b)$$

$$= \left\{ \begin{aligned} & \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right) F^{NP}(b) \\ & \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) \hat{F}^{NP}(b) \end{aligned} \right.$$

$$1. \quad e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) \rightarrow e^{S_j(b_*,Q)} W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right) \quad b_* = \frac{b}{\sqrt{1 + b^2 / b_{\text{max}}^2}}$$

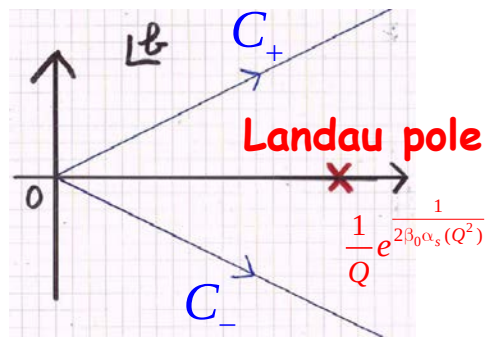
J. Collins, D. Soper,
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C. Balazs, C. Yuan ('00)

$$2. \quad \int_0^\infty db \rightarrow \int_{C_+ + C_-} db$$

$$\left(J_0(bQ_T) = \frac{H_0^{(1)}(bQ_T) + H_0^{(2)}(bQ_T)}{2} \right)$$



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$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) F^{NP}(b)$$

$$= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right) F^{NP}(b) \right. \\ \left. \xrightarrow{\quad} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) \right. \\ \left. \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) \hat{F}^{NP}(b) \right]$$

$$1. e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) \rightarrow e^{S_j(b_*,Q)} W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right) \quad b_* = \frac{b}{\sqrt{1 + b^2 / b_{\text{max}}^2}}$$

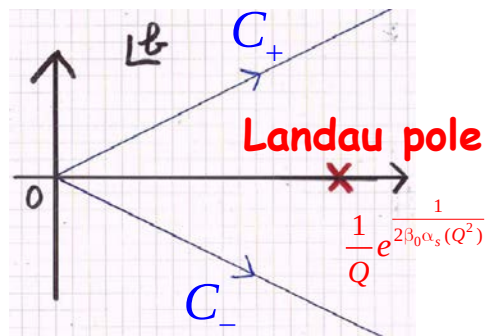
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$$2. \int_0^\infty db \rightarrow \int_{C_+ + C_-} db$$

$$\left(J_0(bQ_T) = \frac{H_0^{(1)}(bQ_T) + H_0^{(2)}(bQ_T)}{2} \right)$$



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$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot \mathbf{Q}_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b, Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*, Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \right. \\
&\quad \left. \xrightarrow{\quad} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) \right] \\
&\quad \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b, Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b)
\end{aligned}$$

$$e^{S_j(b_*, Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) = e^{S_j(b, Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b)$$

$$S_j(b, Q) = - \int_{b_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}$$

$$A_q(\alpha_s) = C_F \frac{\alpha_s}{\pi} + \frac{1}{2} C_F \left\{ \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_G - \frac{5}{9} N_f \right\} \left(\frac{\alpha_s}{\pi} \right)^2 + \dots$$

$$B_q(\alpha_s) = -\frac{3}{2} C_F \frac{\alpha_s}{\pi} + \dots$$

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right) F^{NP}(b) \right. \\
&\quad \left. \xrightarrow{\quad} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) \right] \\
&\quad \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) \hat{F}^{NP}(b)
\end{aligned}$$

$$\hat{F}^{NP}(b) = F^{NP}(b) e^{-\int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}} \frac{W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right)}{W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right)}$$

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right) F^{NP}(b) \right. \\
&\quad \left. \xrightarrow{\quad} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) \right] \\
&\quad \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) \hat{F}^{NP}(b)
\end{aligned}$$

$$\hat{F}^{NP}(b) = F^{NP}(b) e^{-\int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}} \frac{W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right)}{W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right)}$$

accurate around $|b| \sim b_{\text{max}} \sim 1 \text{ GeV}^{-1}$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \frac{1}{4\pi} \int d^2b e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b}) = \int_0^\infty db \frac{b}{2} J_0(bQ_T) \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j(b; \frac{b_0}{b})$$

$$W_j(b; \frac{b_0}{b}) \xrightarrow{b \ll \frac{1}{\Lambda_{\text{QCD}}}} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A})\left(x_A, \frac{b_0^2}{b^2}\right) (C_{\bar{j}k} \otimes f_{k/B})\left(x_B, \frac{b_0^2}{b^2}\right)$$

$$\equiv \bar{P}_{j/A}(x_A, b; b_0) \bar{P}_{\bar{j}/B}(x_B, b; b_0) \quad \text{TMD}$$

$$W_j(b; \frac{b_0}{b}) = W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \frac{W_j(b; \frac{b_0}{b})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \equiv W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{\text{NP}}(b)$$

“primordial k_T ”

$$F^{\text{NP}}(b) = e^{-g_{\text{NP}} b^2}$$

$$g_{\text{NP}} = g_1 + g_2 \ln(Q/2Q_0)$$

1. $e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \rightarrow e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) \quad b_* = \frac{b}{\sqrt{1+b^2/b_{\text{max}}^2}}$

J. Collins, D. Soper, G. Sterman ('85)
A. Kulesza, W. Stirling ('02)
C. Balazs, C. Yuan ('00)

2. $\int_0^\infty db \rightarrow \int_{C_+ + C_-} db$

Landau pole

$\frac{1}{Q} e^{\frac{1}{2\beta_0\alpha_s(Q^2)}}$

$J_0(bQ_T) = \frac{H_0^{(1)}(bQ_T) + H_0^{(2)}(bQ_T)}{2}$

E. Laenen, G. Sterman, W. Vogelsang ('00)
A. Kulesza, G. Sterman, W. Vogelsang ('02)
G. Bozzi, S. Catani, D. de Florian, M. Grazzini ('02)
H. Kawamura, J. Kodaira, KT ('07)

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right) F^{NP}(b) \right] e^{-[g_1+g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Green Arrow}} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{-[\hat{g}_1+\hat{g}_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Yellow Arrow}} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right) \hat{F}^{NP}(b)
\end{aligned}$$

$$\begin{aligned}
\hat{F}^{NP}(b) &= F^{NP}(b) e^{-\int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}} \frac{W_j^{(\text{OPE})}\left(b_*; \frac{b_0}{b_*}\right)}{W_j^{(\text{OPE})}\left(b; \frac{b_0}{b}\right)} \\
&\quad \xleftarrow{\text{Yellow Arrow}} e^{-[g_1+g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xleftarrow{\text{Yellow Arrow}} e^{-[\hat{g}_1+\hat{g}_2 \ln(Q/2Q_0)]b^2}
\end{aligned}$$

accurate around $|b| \sim b_{\text{max}} \sim 1 \text{ GeV}^{-1}$

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \right] e^{-[g_1 + g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Green Arrow}} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Yellow Arrow}} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b)
\end{aligned}$$

$$\begin{aligned}
\hat{F}^{NP}(b) &= F^{NP}(b) e^{-\int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}} \frac{W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})} \\
&\quad \xleftarrow{\text{Yellow Arrow}} e^{-[g_1 + g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xleftarrow{\text{Yellow Arrow}} e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2}
\end{aligned}$$

accurate around $|b| \sim b_{\text{max}} \sim 1 \text{ GeV}^{-1}$

$$W_j^{(\text{OPE})}(b; \frac{b_0}{b}) = (C_{ji} \otimes f_{i/A}) \left(x_A, \frac{b_0^2}{b^2} \right) (C_{\bar{j}k} \otimes f_{k/B}) \left(x_B, \frac{b_0^2}{b^2} \right)$$

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \right] e^{-[g_1 + g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Green Arrow}} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Yellow Arrow}} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b)
\end{aligned}$$

$$\hat{g}_1 = g_1 + \frac{1}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{4Q_0^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\} + \ln \left[W_j^{(\text{OPE})}(b) / W_j^{(\text{OPE})}(b_*) \right]$$

$$\hat{g}_2 = g_2 + \frac{2}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} A_j(\alpha_s(\mu^2))$$

accurate around $|b| \sim b_{\text{max}} \sim 1 \text{ GeV}^{-1}$

$$b_* = \frac{b}{\sqrt{1 + b^2 / b_{\text{max}}^2}}$$

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \right] e^{-[g_1 + g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Green Arrow}} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Yellow Arrow}} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b)
\end{aligned}$$

$$\begin{aligned}
\hat{g}_1 &= g_1 + \frac{1}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{4Q_0^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\} + \ln \left[W_j^{(\text{OPE})}(b) / W_j^{(\text{OPE})}(b_*) \right] \\
&\simeq g_1 - \frac{1}{b_{\text{max}}^2} \left\{ 2A_j(\alpha_s(b_0^2/b_{\text{max}}^2)) \ln \frac{2Q_0 b_{\text{max}}}{b_0} + B_j(\alpha_s(b_0^2/b_{\text{max}}^2)) \right\} + \dots \\
\hat{g}_2 &= g_2 + \frac{2}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} A_j(\alpha_s(\mu^2)) \simeq g_2 - \frac{2}{b_{\text{max}}^2} A_j(\alpha_s(b_0^2/b_{\text{max}}^2))
\end{aligned}$$

accurate around $|b| \sim b_{\text{max}} \sim 1 \text{ GeV}^{-1}$

$$b_* = \frac{b}{\sqrt{1 + b^2 / b_{\text{max}}^2}}$$

$$S_j(b,Q)=-\int_{b_0^2/b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left\{\left(\ln\frac{Q^2}{\mu^2}\right)A_j\left(\alpha_s(\mu^2)\right)+B_j\left(\alpha_s(\mu^2)\right)\right\}$$

$$A_j\left(\alpha_s\right)=A_j^{(1)}\frac{\alpha_s}{\pi}+A_j^{(2)}\left(\frac{\alpha_s}{\pi}\right)^2+A_j^{(3)}\left(\frac{\alpha_s}{\pi}\right)^3+\cdots\qquad A_j^{(n)}>0$$

$$\simeq 1.33\frac{\alpha_s}{\pi}+2.67\left(\frac{\alpha_s}{\pi}\right)^2+6.71\left(\frac{\alpha_s}{\pi}\right)^3+\cdots$$

$$\hat{g}_2\simeq g_2-\frac{2}{b_{\max}^2}A_j\left(\alpha_s(b_0^2/b_{\max}^2)\right)$$

$$\simeq g_2-0.5\,\mathrm{GeV}^2\qquad\left(b_{\max}\sim 1\,\mathrm{GeV}^{-1}\right)$$

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \right] e^{-[g_1 + g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Green Arrow}} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Yellow Arrow}} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b)
\end{aligned}$$

$$\begin{aligned}
\hat{g}_1 &= g_1 + \frac{1}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{4Q_0^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\} + \ln \left[W_j^{(\text{OPE})}(b) / W_j^{(\text{OPE})}(b_*) \right] \\
&\simeq g_1 - \frac{1}{b_{\text{max}}^2} \left\{ 2A_j(\alpha_s(b_0^2/b_{\text{max}}^2)) \ln \frac{2Q_0 b_{\text{max}}}{b_0} + B_j(\alpha_s(b_0^2/b_{\text{max}}^2)) \right\} + \dots \\
\hat{g}_2 &= g_2 + \frac{2}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} A_j(\alpha_s(\mu^2)) \simeq g_2 - \frac{2}{b_{\text{max}}^2} A_j(\alpha_s(b_0^2/b_{\text{max}}^2))
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} &= \frac{1}{4\pi} \int d^2b \, e^{ib \cdot Q_T} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) F^{NP}(b) \\
&= \left[\sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*,Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \right] e^{-[g_1 + g_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Green Arrow}} \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2} \\
&\quad \xrightarrow{\text{Yellow Arrow}} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b)
\end{aligned}$$

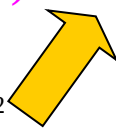
$$\begin{aligned}
\hat{g}_1 &= g_1 + \frac{1}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{4Q_0^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\} + \ln \left[W_j^{(\text{OPE})}(b) / W_j^{(\text{OPE})}(b_*) \right] \\
&\simeq g_1 - \frac{1}{b_{\text{max}}^2} \left\{ 2A_j(\alpha_s(b_0^2/b_{\text{max}}^2)) \ln \frac{2Q_0 b_{\text{max}}}{b_0} + B_j(\alpha_s(b_0^2/b_{\text{max}}^2)) \right\} + \dots \\
\hat{g}_2 &= g_2 + \frac{2}{b^2} \int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} A_j(\alpha_s(\mu^2)) \simeq g_2 - \frac{2}{b_{\text{max}}^2} A_j(\alpha_s(b_0^2/b_{\text{max}}^2))
\end{aligned}$$

$$\hat{g}_2 < g_2 \quad !!$$

$$\frac{d\sigma}{dQ^2 dQ_T^2 dy} = \frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} + \frac{d\sigma^{(\text{fin})}}{dQ^2 dQ_T^2 dy}$$

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \left\{ \begin{array}{l} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*, Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \\ \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b, Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b) \end{array} \right.$$

$e^{-[g_1 + g_2 \ln(Q/2Q_0)]b^2}$

 $e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2}$


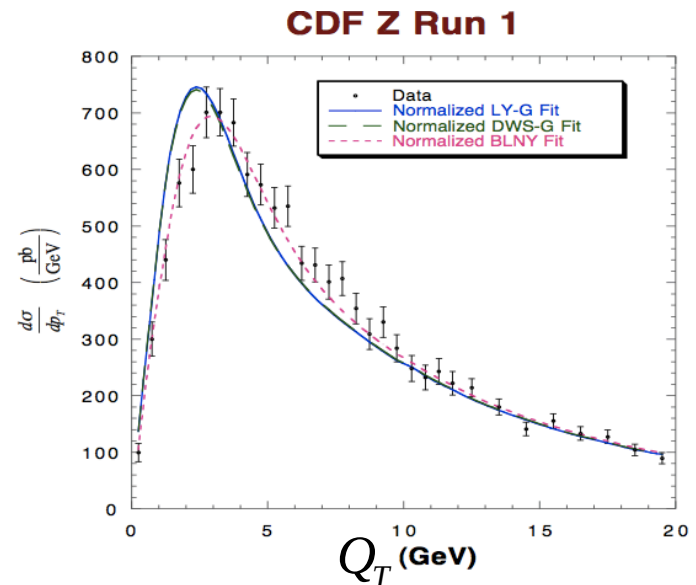
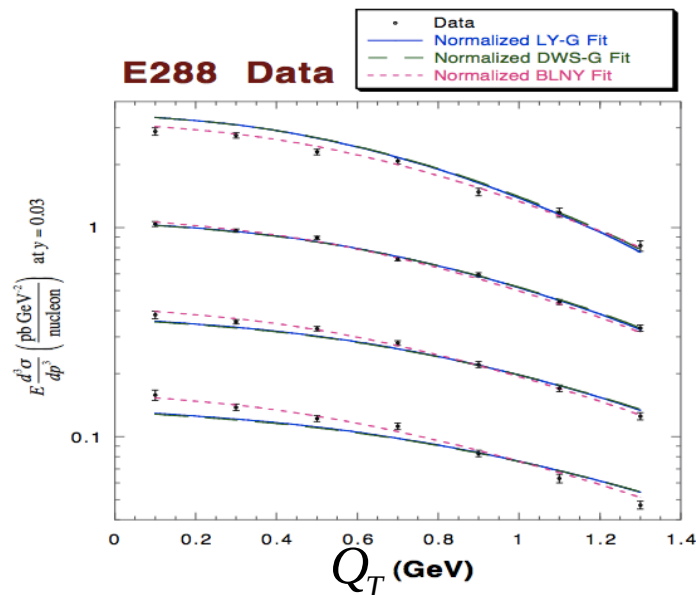
1. $b_* = b / \sqrt{1 + b^2 / b_{\text{max}}^2}$

Global fit $g_1 = -0.08 \text{ GeV}^2, g_2 = 0.67 \text{ GeV}^2$ ($Q_0 = 1.6 \text{ GeV}$)
A. Kulesza, W. Stirling ('03)

$(b_{\text{max}} = 0.5 \text{ GeV}^{-1})$ $g_1 = 0.016 \text{ GeV}^2, g_2 = 0.54 \text{ GeV}^2$ ($Q_0 = 1.6 \text{ GeV}$)
F. Landry, R. Brock, P. Nadolsky. C. Yuan ('03)

Fit to Drell-Yan and Z boson production

$$F^{NP}(b) = e^{-[g_1 + g_2 \ln(Q/2Q_0) + g_1 g_3 \ln(100x_A x_B)]} b^2$$



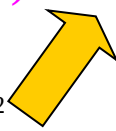
g_1	0.21	0.016
g_2	0.68	0.54
g_3	-0.60	0.00

$$b_{\max} = 0.5 \text{ GeV}^{-1}$$

$$\frac{d\sigma}{dQ^2 dQ_T^2 dy} = \frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} + \frac{d\sigma^{(\text{fin})}}{dQ^2 dQ_T^2 dy}$$

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Global fit

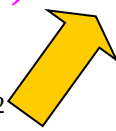
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2. $\int_0^\infty db \rightarrow \int_{C_+ + C_-} db$

$\hat{g}_2 \simeq g_2 - \frac{2}{b_{\text{max}}^2} A_j \left(\alpha_s(b_0^2 / b_{\text{max}}^2) \right)$

$$S_j(b,Q)=-\int_{b_0^2/b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left\{\left(\ln\frac{Q^2}{\mu^2}\right)A_j\left(\alpha_s(\mu^2)\right)+B_j\left(\alpha_s(\mu^2)\right)\right\}$$

$$A_j\left(\alpha_s\right)=A_j^{(1)}\frac{\alpha_s}{\pi}+A_j^{(2)}\left(\frac{\alpha_s}{\pi}\right)^2+A_j^{(3)}\left(\frac{\alpha_s}{\pi}\right)^3+\cdots\qquad A_j^{(n)}>0$$

$$\simeq 1.33\frac{\alpha_s}{\pi}+2.67\left(\frac{\alpha_s}{\pi}\right)^2+6.71\left(\frac{\alpha_s}{\pi}\right)^3+\cdots$$

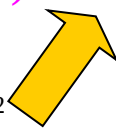
$$\hat{g}_2\simeq g_2-\frac{2}{b_{\max}^2}A_j\left(\alpha_s(b_0^2/b_{\max}^2)\right)$$

$$\simeq g_2-0.5\,\mathrm{GeV}^2\qquad\left(b_{\max}\sim 1\,\mathrm{GeV}^{-1}\right)$$

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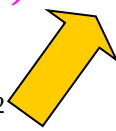
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" revised b_* " $g_2 \simeq 0.19 \text{ GeV}^2$
 $(b_{\text{max}} = 1.5 \text{ GeV}^{-1})$ A. Konychev, P. Nadolsky ('06)

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Global fit at NLL using “complex b” approach

M. Hirai, H. Kawamura, KT

Experimental data sets

	Exp	$\sqrt{s}$ (GeV)	Target	Q_T -range (GeV)	Q range (GeV)	# of data ($p_T < 22$ GeV)
DY	R209	62	P-P	0.2 – 1.8	5.0 - 8.0	5
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Z^0	CDF run-0	1800	P-Pbar	0.0 – 22.8	75 - 105	7
	CDF run-1	1800	P-Pbar	0.0 – 22.0	66 - 116	33
	D0 run-1	1800	P-Pbar	0.0 – 22.0	75 - 105	15

$$\frac{d\sigma}{dQ^2 dQ_T^2} = \frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2} + \frac{d\sigma^{(\text{fin})}}{dQ^2 dQ_T^2}$$

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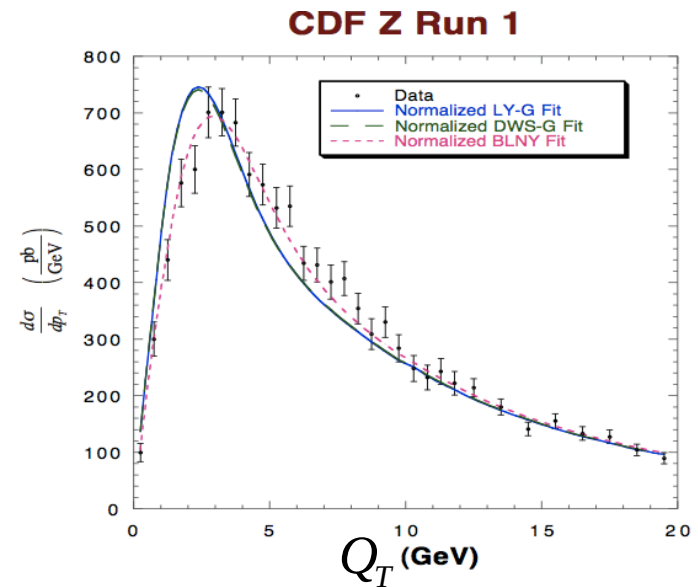
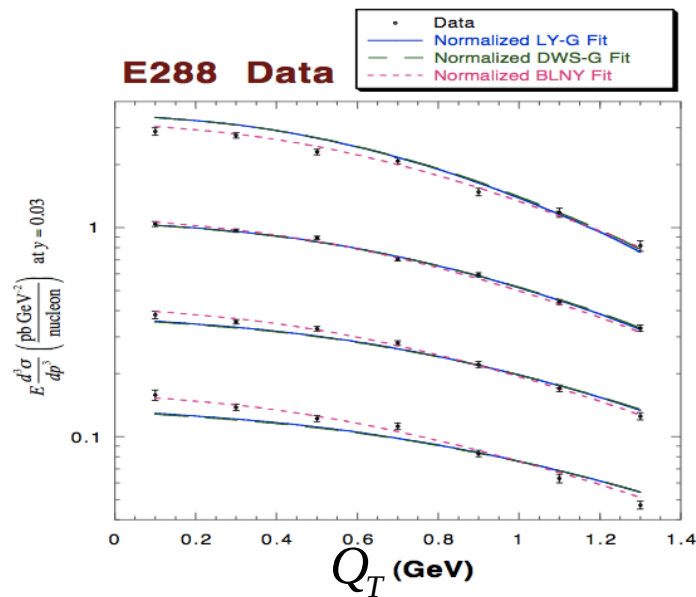
2-parameter fit of $\hat{F}^{NP}(b) = e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2}$



determine $\hat{g}_1, \hat{g}_2$

Fit to Drell-Yan and Z boson production

$$F^{NP}(b) = e^{-[g_1 + g_2 \ln(Q/2Q_0) + g_1 g_3 \ln(100x_A x_B)]} b^2$$



g_1	0.21	0.016
g_2	0.68	0.54
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$$b_{\max} = 0.5 \text{ GeV}^{-1}$$

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2-parameter fit of $\hat{F}^{NP}(b) = e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]b^2}$

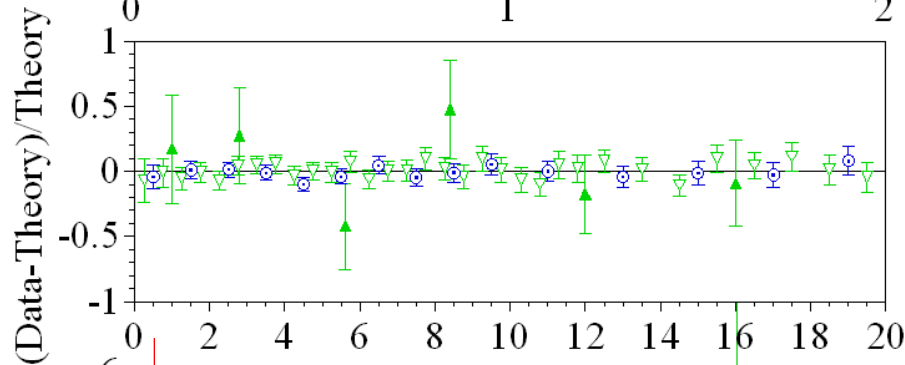
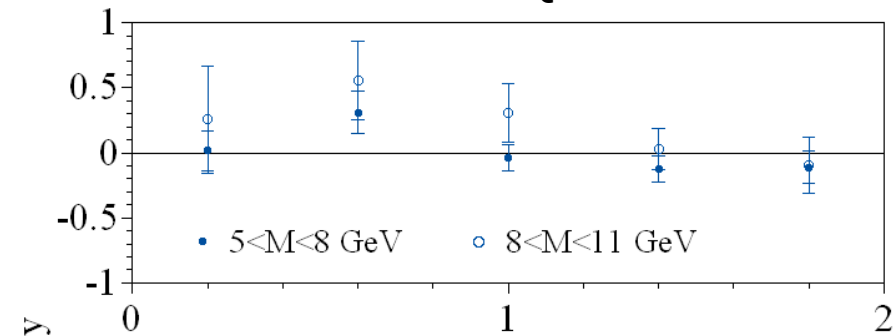


determine $\hat{g}_1, \hat{g}_2$

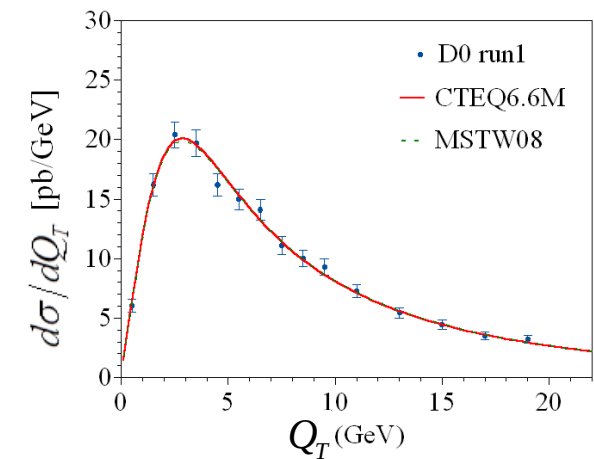
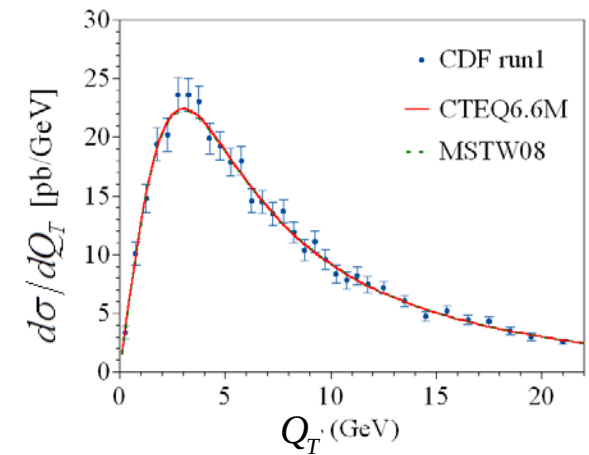
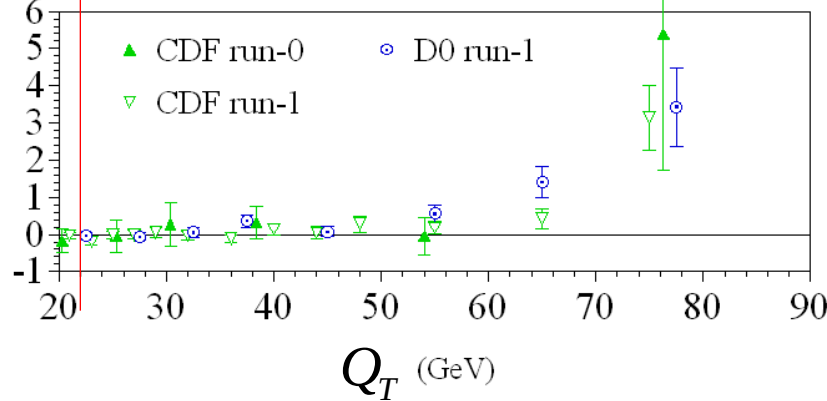
Data vs. Theory

R209

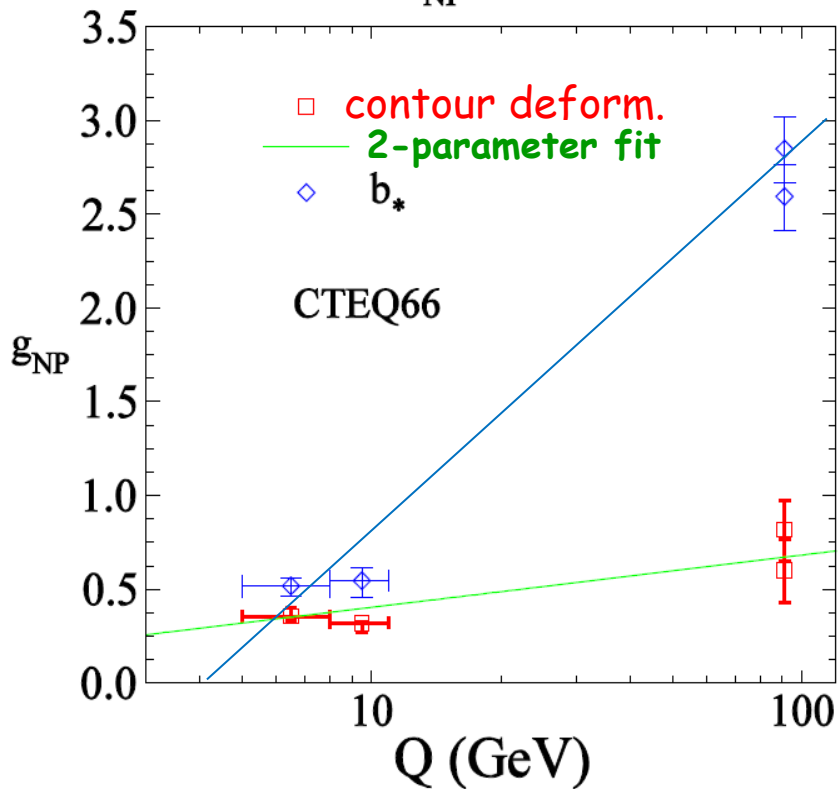
CTEQ6.6



D0, CDF

 $Q_T < 22 \text{ GeV}$ 

g_{NP} vs. Q



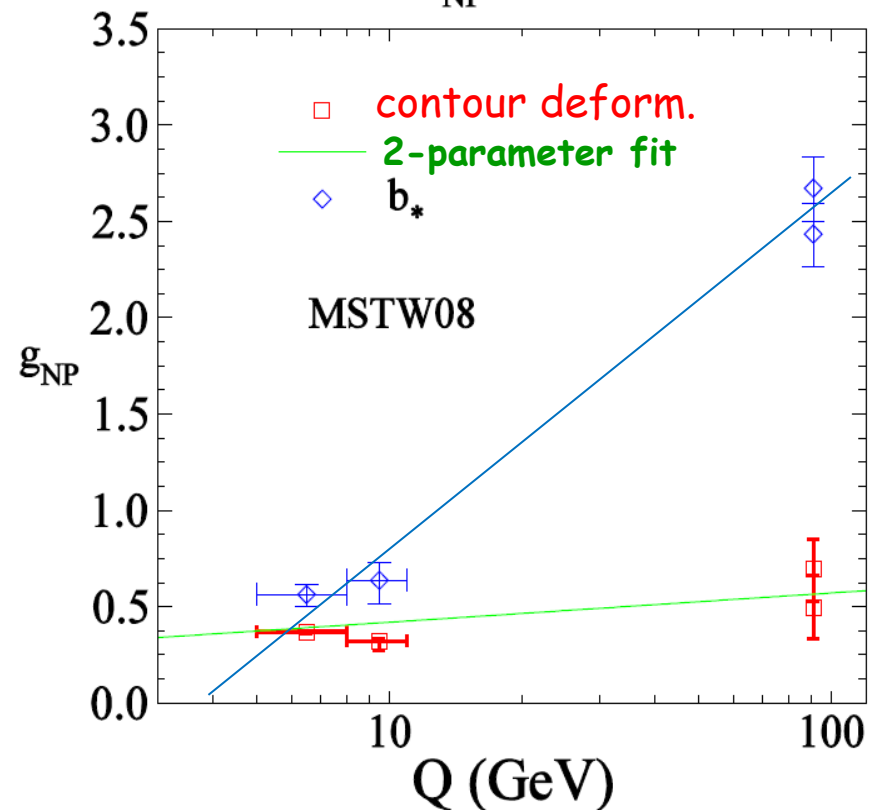
$$\hat{g}_1 = 0.241^{+0.026}_{-0.028} \text{ GeV}^2$$

$$\hat{g}_2 = 0.121^{+0.041}_{-0.038} \text{ GeV}^2$$

$$\hat{g}_{NP} = \hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)$$

($Q_0 = 1.3 \text{ GeV}$)

g_{NP} vs. Q Hirai, Kawamura, KT

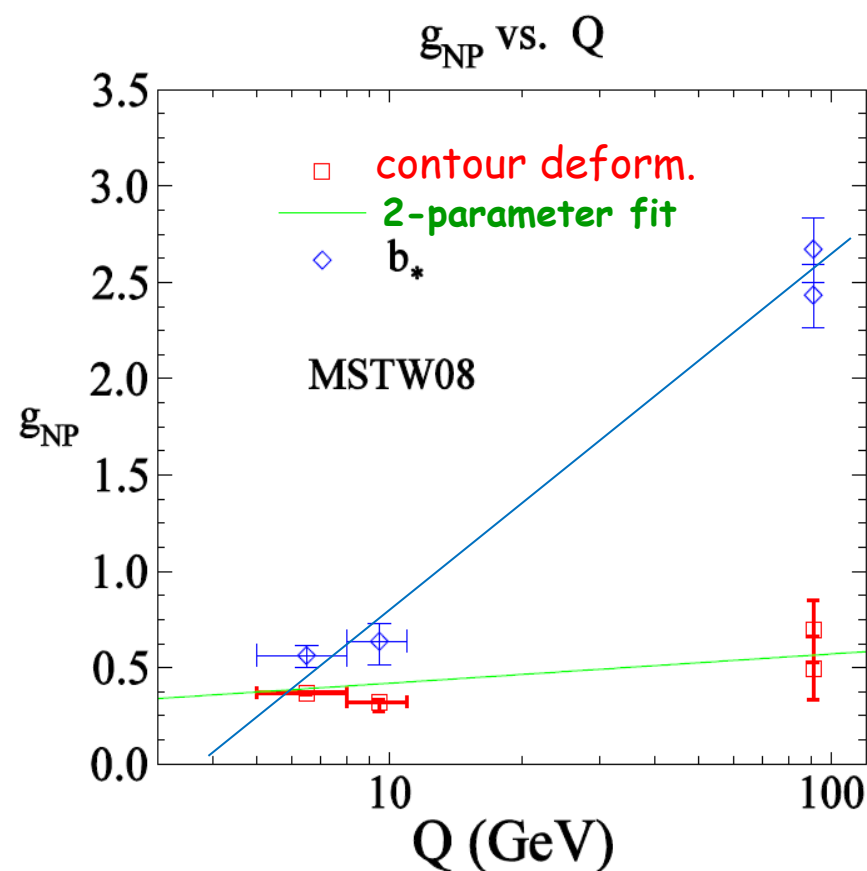
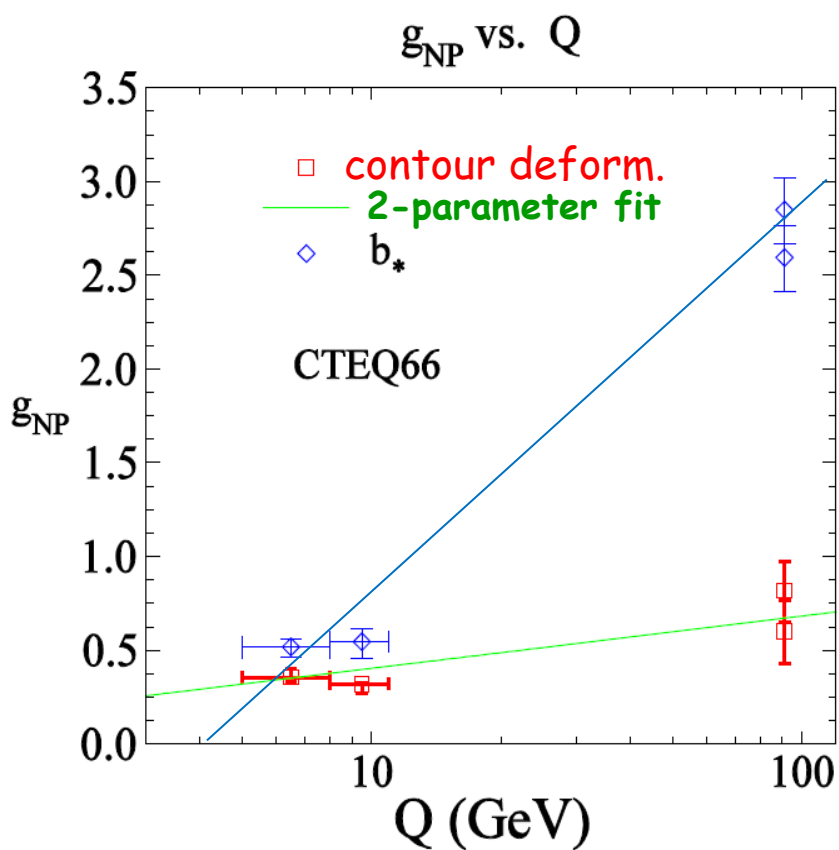


$$\hat{g}_1 = 0.330^{+0.024}_{-0.026} \text{ GeV}^2$$

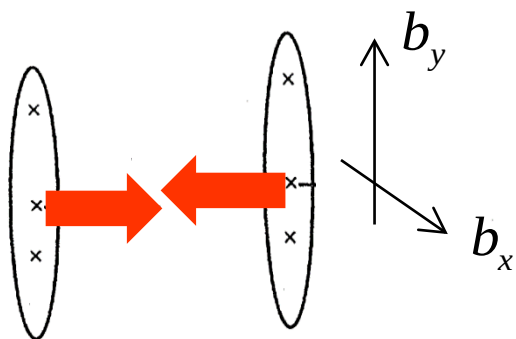
$$\hat{g}_2 = 0.066^{+0.039}_{-0.037} \text{ GeV}^2$$

$$\left[g_1 = 0.016 \text{ GeV}^2, g_2 = 0.54 \text{ GeV}^2 \right]$$

F. Landry, R. Brock, P. Nadolsky. C. Yuan ('03)



$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2} = \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b,Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{\text{NP}}(b)$$



$$\hat{F}^{\text{NP}}(b) = e^{-\hat{g}_{\text{NP}} b^2} \quad (\hat{g}_{\text{NP}} = \hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0))$$

$$e^{-\frac{k_T^2}{4\hat{g}_{\text{NP}}}}$$

$$\langle k_T^2 \rangle = 4\hat{g}_{\text{NP}} \lesssim 2 \text{ GeV}^2$$

$$\langle k_T^2 \rangle_{1\text{-proton}} \lesssim 1 \text{ GeV}^2$$

Summary: QCD resummation for Q_T distribution

" b_* " vs. "complex b "

$$\frac{d\sigma^{(\text{res})}}{dQ^2 dQ_T^2 dy} = \left\{ \begin{array}{l} \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \int_0^\infty db \frac{b}{2} J_0(bQ_T) e^{S_j(b_*, Q)} W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*}) F^{NP}(b) \\ \sum_j \hat{\sigma}_{j\bar{j}}(Q^2) \left(\int_{C_+} db \frac{b}{4} H_0^{(1)}(bQ_T) + \int_{C_-} db \frac{b}{4} H_0^{(2)}(bQ_T) \right) e^{S_j(b, Q)} W_j^{(\text{OPE})}(b; \frac{b_0}{b}) \hat{F}^{NP}(b) \end{array} \right.$$

$$\hat{F}^{NP}(b) = F^{NP}(b) e^{-\int_{b_0^2/b_*^2}^{b_0^2/b^2} \frac{d\mu^2}{\mu^2} \left\{ \left(\ln \frac{Q^2}{\mu^2} \right) A_j(\alpha_s(\mu^2)) + B_j(\alpha_s(\mu^2)) \right\}} \frac{W_j^{(\text{OPE})}(b_*; \frac{b_0}{b_*})}{W_j^{(\text{OPE})}(b; \frac{b_0}{b})}$$

$$\hat{F}^{NP}(b) = e^{-[\hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0)]} b^2$$

$$\hat{g}_2 \simeq g_2 - \frac{2}{b_{\text{max}}^2} A_j(\alpha_s(b_0^2/b_{\text{max}}^2))$$

global fit to experimental data

$$\hat{g}_1 = 0.241_{-0.028}^{+0.026} \text{ GeV}^2 \quad \hat{g}_2 = 0.121_{-0.038}^{+0.041} \text{ GeV}^2 \quad \text{CTEQ6.6}$$

$$\hat{g}_1 = 0.330_{-0.026}^{+0.024} \text{ GeV}^2 \quad \hat{g}_2 = 0.066_{-0.037}^{+0.039} \text{ GeV}^2 \quad \text{MSTW08}$$

$$\Leftrightarrow \left[\begin{array}{l} g_1 = 0.016 \text{ GeV}^2, \quad g_2 = 0.54 \text{ GeV}^2 (b_{\text{max}} = 0.5 \text{ GeV}^{-1}) \\ \text{F. Landry, R. Brock, P. Nadolsky, C. Yuan ('03)} \\ \text{"revised } b_*" (b_{\text{max}} = 1.5 \text{ GeV}^{-1}) \quad g_2 \simeq 0.19 \text{ GeV}^2 \\ \text{A. Konychev, P. Nadolsky ('06)} \end{array} \right]$$

$$\langle k_T^2 \rangle_{1\text{-proton}} = 2 \{ \hat{g}_1 + \hat{g}_2 \ln(Q/2Q_0) \} \lesssim 1 \text{ GeV}^2$$

