A coordinate description of partonic processs

Radcor-Loopfest

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A look at perturbative long distance behavior of amplitudes and cross sections in coordinate space. “Where to large corrections come from?

– Coordinate space leading regions for fixed-angle scattering amplitudes
– Approximations and factorization
– Wilson lines: webs and the running coupling in coordinate space
– Extension to cross sections
– Cancellation: the largest time equation

Coordinate space leading regions..

[Analogs in momentum space for amplitudes and cross sections are well-known and continue to be studied (Collins, GS 1981; Sen, 1983; ... Feige & Schwartz 1403.6472; Caron-Huot 1501.0354)]

Here, look at coordinate space VEVs for all massless fields in configurations reflecting scattering. The scalar propagator, for example, with $D = 2 - 2\varepsilon$:

$$\Delta(y-x) \;=\; \frac{1}{4\pi^{2-\varepsilon}}\,\frac{1}{\left(-(y-x)^2+i\epsilon\right)^{1-\varepsilon}}$$

Schematically, we have integrals over positions of internal vertices y_k:

$$G_N(x_1,\dots,x_N) \;=\; \langle 0|T\left(\phi_N(x_N)\,\cdots\,\phi_1(x_1)\right)|0\rangle$$

$$\sum_{\substack{\text{many}\\ \text{terms}}}\;\prod_{\substack{\text{some}\\ \text{vertices }k}}\int d^D y_k \prod_{\text{lines }j}\frac{\text{numerator}}{\left[-(\Sigma_{k'}\,\eta_{jk'}\,y_{k'}+\Sigma_{k'}\,\eta_{jl}\,x_l)^2+i\epsilon\right]^{p_j}}$$

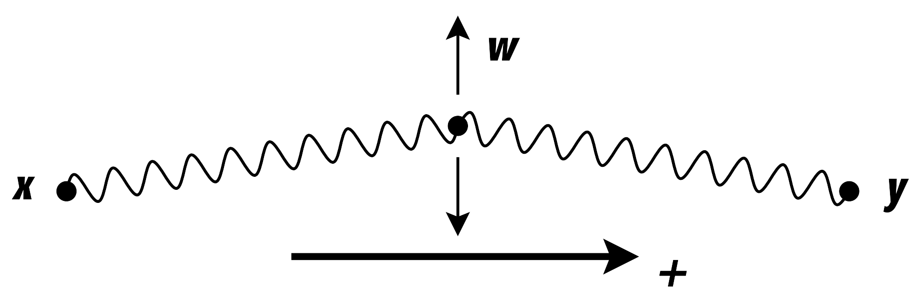
Powers $p_j = 1-\varepsilon$ (boson) or $2-\varepsilon$ (fermion, or derivative of boson).

Once UV renormalized, G_N is singular only at pinches in the complex integrals over positions of vertices, y_k between "incoming" and "outgoing" propagators (on the light cone).
(S. Date 1983; A.O. Erdogan 1312.0058, PRD).

Example:

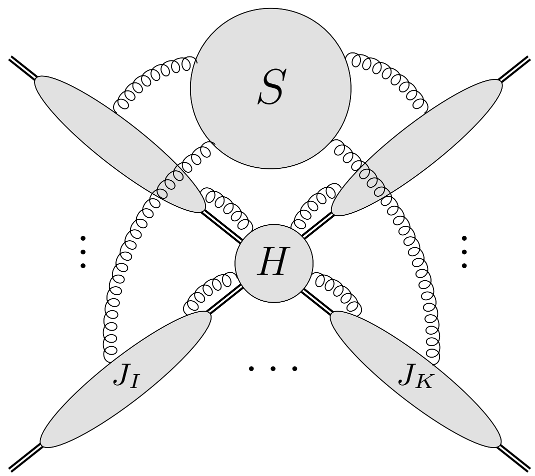
$$\int d^D w \; \frac{1}{\left(-2(y-w)^+(y-w)^- - (\mathbf{y}_\perp - \mathbf{w}_\perp)^2 + \mathrm{i}\epsilon\right)^{1-\varepsilon}}$$
$$\times \frac{1}{\left(-2(w-x)^+(w-x)^- - (\mathbf{w}_\perp - \mathbf{x}_\perp)^2 + \mathrm{i}\epsilon\right)^{1-\varepsilon}}$$

Corresponding to a pinch at $w^-, \mathrm{w}_\perp = 0$ when x and y are lined up in the $+$ direction:



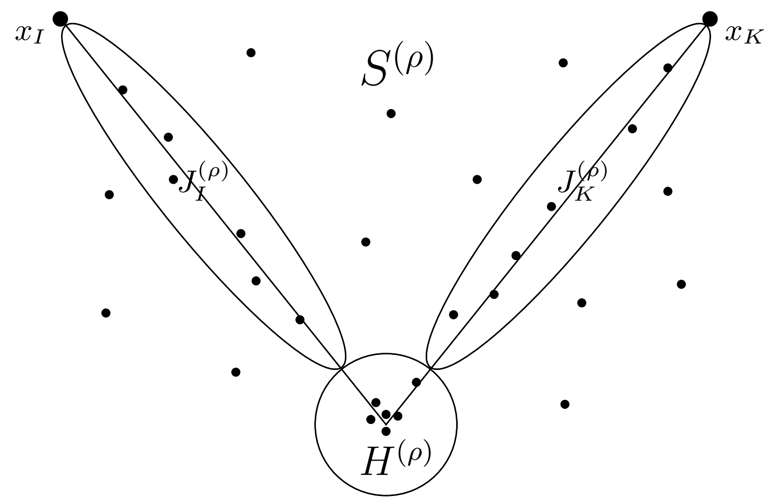
The result:

The general "leading (-power) region"; as in momentum space, a "physical picture":



General pinch surface (ρ) in coordinate space. Jets are in directions β_I from the position of a hard scattering. Each $\beta_I^2 \to 0 = \bar{\beta}_I^2$, $\beta_I \cdot \bar{\beta}_I = 1$. Vertices group along the β_I, near the origin, or are at finite distances from these.

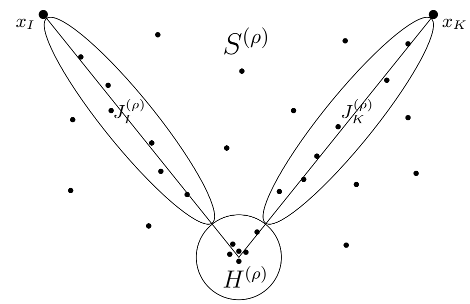
A picture of "where the vertices are on pinch surface ρ":



- **Vertices in $H^{(\rho)}$ are near the origin**
- **Vertices in $J_I^{(\rho)}$ are "near" rays $\beta_I^\mu \propto x_I^\mu$ for $x_I^2 \to 0$**
- **Vertices in $S^{(\rho)}$ are separated from the origin and the rays.**

This organizes large numbers of diagrams related by connecting vertices in all possible ways.

In each such region (ρ), introduce an approximation operator acting on each diagram γ:
(GS, Erdogan 1411.4588)



$$
\begin{aligned}
t_\rho \gamma^{(n)} &= \gamma^{(n)}\Big|_{\mathrm{div}\ n[\rho]} \\
&\equiv \prod_I \int d\tau^{(I)}\ S^{(\rho)}_{\{\mu_I\}}(\{\tau^{(I)}\})\ \beta_I^{\mu_I}\bar\beta_{I,\mu'_I} \\
&\quad \times \int d\eta^{(I)} \int d^{D-1}z^{(I)}\ J_I^{(\rho)\mu'_I\nu'_I}(z^{(I)},\eta^{(I)})\ \bar\beta_{I,\nu'_I}\beta_I^{\nu_I} \int d^{D-1}y^{(I)}\ H^{(\rho)}_{\{\nu_I\}}(y^{(I)})
\end{aligned}
$$

For gluons attaching "soft" function to jet I in direction β_I, keep only the $\bar\beta_I$ polarization and the coordinate τ_I along the β_I direction:

This can be a starting point for "deriving" Soft-Collinear EFT: in J_I:
$A^\mu = A_c^\mu(x) + A_s^\mu(\bar\beta_I \cdot x)$. (E.g., as described in Becher, Broggio, Ferroglia 2014. Stewart talk here.)

The operators t_ρ organize all divergences as external points approach the light cone relative to the hard scattering:

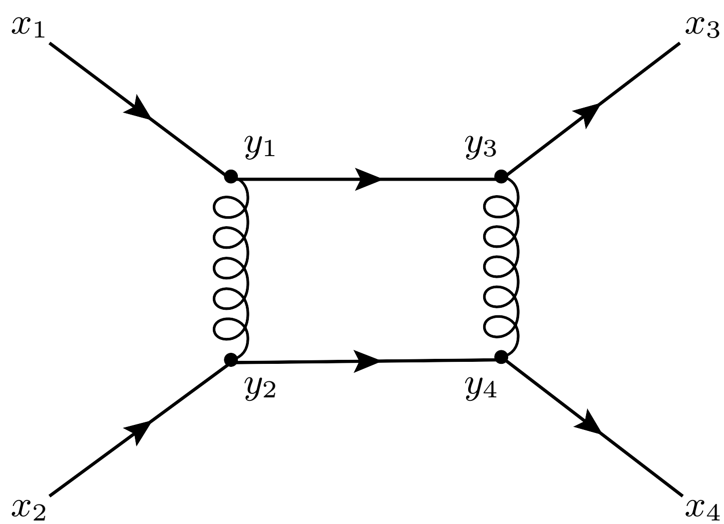
$$\boldsymbol{R}_P^{(n)}\,\gamma^{(n)}|_{\mathrm{div}\ \hat{n}[\rho]} \;=\; \left[\gamma^{(n)} \;+\; \sum_{N\in\mathcal{N}_P[\gamma^{(n)}]}\;\prod_{\rho\in N}\,(\,-\,t_\rho)\,\gamma^{(n)}\right]\Big|_{\mathrm{div}\ \hat{n}[\rho]} \;=\; 0$$

Each t_ρ acts to approximate the integrand by its leading behavior near the singular surface. Following Collins and Soper (1983) in axial gauge and Collins (2013) in covariant gauge, the sum is over all possible nested regions, which cancels overlapping divergences. For this process formalizes a strategy of regions. (Beneke Smirnov 1998, Jantzen 2011.) Within each region ρ, only t_ρ approximation contributes, but each approximation extends over all coordinate space.

This generalizes arguments given for Sudakov-related processes. The arguments are applicable to momentum space, and the relation to NNLO arguments (c.f. talk by M. Czakon) is clear. Each subtraction corresponds to a leading region. Any application illustrates that the general elimination of double counting can be complicated even at low orders. Complicated or not, double counting can be avoided to all orders, even with jets in the final state.

As $x_I^2 \to 0$ relative to the hard scattering (H): $x_I^\mu \propto \beta_I^\mu$, $\beta_I^2 = 0$, this allows the derivation of a factorized amplitude in coordinate space

Already at one loop, nesting for fixed-angle scattering becomes nontrivial because in QCD, hard scatterings can be disjoint



Either gluon can carry the hard scattering, with the other soft or (2 choices of) collinear (or part of the hard scattering).

The result for VEVs with fixed-angle geometry:

$$G\left(\{x_I\}\right) \;=\; \prod_{I=1}^{a} \int d\eta_I \; j_I^{\rm part}(x_I, \eta_I\bar{\beta}_I)\; S_{\rm ren}(\{\beta_I \cdot \beta_J\}) \;\; \mathcal{H}\left(\{\eta_I\bar{\beta}_I\}\right)$$

with "jet" functions

$$j_I^{\,{\rm part}[{\rm f}_\phi]}(x_I, \eta_I\bar{\beta}_I) \;=\; c_I^{\rm cusp}(\beta_I, \bar{\beta}_I)\left\langle 0 \left| T\Big(\phi(x_I)\, \phi^\dagger(\eta_I\bar{\beta}_I)\Phi_{\bar{\beta}_I}^{[f_\phi]\,\dagger}(\infty, \eta_I\bar{\beta}_I)\Big) \right| 0 \right\rangle ,$$

and a soft function constructed entirely from Wilson lines in the β_I directions.

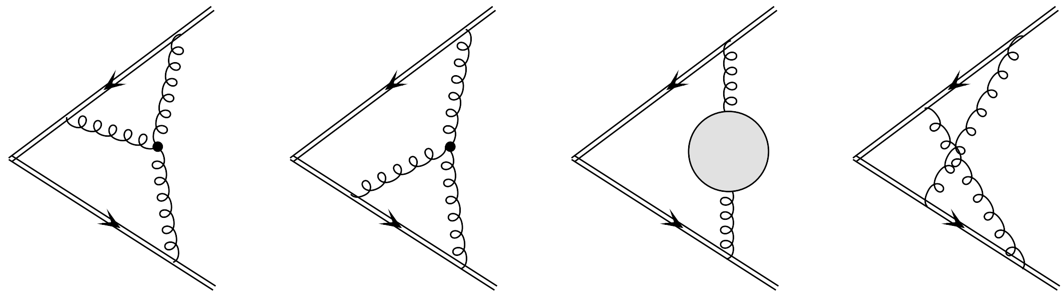
Coordinate picture of soft radiation (cusps and polygons in QCD)

(Erdogan (...1312.3310, PRD), Mitov, GS, Sung, PRD,)

- **Combinatoric exponentiation in coordinate space (It's more general, but we'll consider just the cusp.)** (Gardi, Magnea – see talks here)
- **Will find an interesting 'geometrical' interpretation directly in QCD, which becomes exact for large-N.**
- **The 2-line eikonal form factor is the exponential of a sum of two-eikonal irreducible diagrams, the "webs" with modified color factors:**
  (Gatheral, Frenkel Taylor, GS, 1981-83)

$$\boldsymbol{A} = \exp\left[\sum_{i=1}^{\infty} \boldsymbol{w}^{(i)}\right]$$

- **"Webs" in the exponent, $w^{(i)}$. are 2-eikonal irreducible diagrams. At 2 loops:**



- **All have color factor (fundamental representation) $C_F C_A$ (only – no C_F^2).**

- **Here's how it works. Say we know the exponent $w^{(i)}$ to order N. Expand to $N+1$st order, as a sum of diagrams and in terms of the exponential**

$$\boldsymbol{A^{(N+1)} = \left(\exp \left[\sum_{i=1}^{N+1} w^{(i)} \right] \right)^{(N+1)}}$$

$$\boldsymbol{A^{(N+1)} = \sum_{D^{(N+1)}} D^{(N+1)} \,.}$$

- **This gives a formula for the highest order in the exponent:**

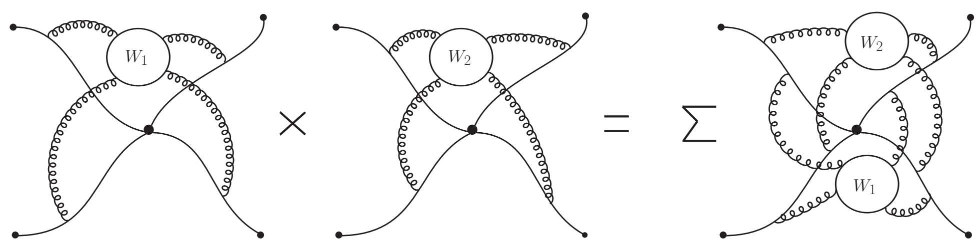
$$\boldsymbol{w^{(N+1)} = \sum_{D^{(N+1)}} D^{(N+1)} - \left[\sum_{m=2}^{N+1} \frac{1}{m!} \sum_{i_m=1}^{N} \dots \sum_{i_1=1}^{N} w^{(i_m)}\, w^{(i_{m-1})} \dots w^{(i_1)} \right]^{(N+1)}}$$

- **The simplest (and most general) proofs are in coordinate spacee.**

To the relation

$$\boldsymbol{w}^{(N+1)} = \sum_{D^{(N+1)}} \boldsymbol{D}^{(N+1)} - \Big[\sum_{m=2}^{N+1} \frac{1}{m!} \sum_{i_m=1}^{N} \ldots \sum_{i_1=1}^{N} \boldsymbol{w}^{(i_m)}\, \boldsymbol{w}^{(i_{m-1})} \ldots \boldsymbol{w}^{(i_1)} \Big]^{(N+1)}$$

- Now apply the 'abelian' graphical identity, which holds for any number, length or shape of Wilson lines, simply an expression of the path ordering of $\exp[\int d\lambda \beta \cdot A(\beta\lambda)]$:



The ("shuffle") identity allows us to interpret the products of order w_n's, $n = 1 \ldots N$ in terms of $N+1$st order diagrams, so that the effect of all the lower orders is to modify color factors, since these are unaffected by the identity.

- **Important: The webs automatically subtract all divergences where there is more than one 'subjet' in the web. The entire web is either hard, soft, or collinear to one line or the other.** (Erdogan, GS, 1112.4564 and 1411.4588, PRD)

- **The web acts like a single gluon (consistent with the dipole exponentiation).**

- **As ordered exponentials, the webs can be constructed in coordinate space. Care must be taken to preserve gauge invariance, or double-logarithmic exponentiation fails in general. With this in mind, the result, with an IR cutoff L, is:**

$$E(L,\varepsilon) = \int_0^L \frac{d\lambda}{\lambda} \frac{d\sigma}{\sigma} w\left(\alpha_s\left(1/\lambda\sigma\right)\right)$$

- **Here σ and λ are distances along the eikonal lines. The invariant size of the web fixes the running coupling.**

$$w = -\frac{1}{4}A(\alpha_s) + \mathcal{O}(\varepsilon)$$

- Again,

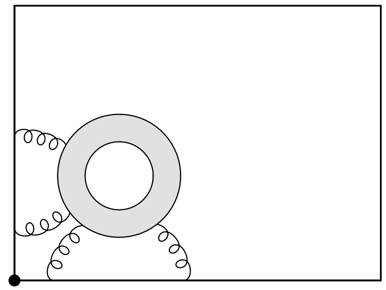
$$E(L,\varepsilon) = \int_0^L \frac{d\lambda}{\lambda}\frac{d\sigma}{\sigma}\, w\left(\alpha_s\left(1/\lambda\sigma\right)\right)$$

- A “surface” interpretation is tempting, viz. minimizing a 5-dimensional surface (Alday, Maldacena 2007).

  In QCD the coupling runs as the integral passes over the surface.

  The result holds to all orders in perturbation theory, and gives the structure of power corrections by Borel or related analysis. It is not limited to “dressed gluon” or related subsets of diagrams.

- **Another interesting correspondence: polygonal Wilson lines** (Korchemskaya, Korchemsky (1993), Drummond et al (2008). . .). **Webs appear in "corners, and when defined in a gauge-invariant fashion give the leading singularities:**



- **Neglecting the running of α_s, another analogy to minimal surfaces in 5-D:**

$$\sum_{a=1}^{4} W_a(\beta_a, \beta'_a) \; = \; \int_{-1}^{1} dy_1 \int_{-1}^{1} dy_2 \, \frac{4w_{\rm conformal}}{(1 - y_1^2)(1 - y_2^2)}$$

(Again, as in Alday, Maldacena 2007)

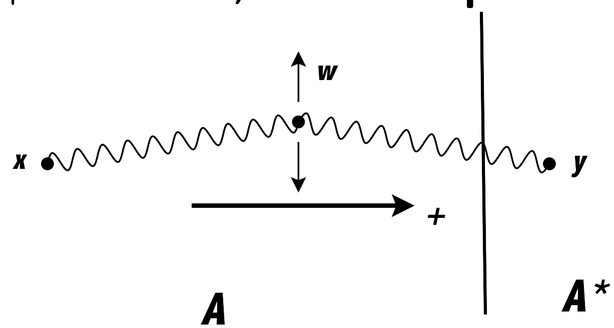
A coordinate picture for cross section: Schematically: $\Pi_k \int d^D k A^*(\{k\}) \, \Pi_k \, \delta_+(k^2) A(\{k\})$

Start with the "cut propagator" with momentum flowing out of w and into y:

$$\int \frac{d^D k}{(2\pi)^D} \, \mathrm{e}^{-ik\cdot(y-w)} \, (2\pi) \, \delta_+\left(k^2\right) \;=\; \frac{1}{4\pi^{2-\varepsilon}} \, \frac{1}{(-(y-w)^2 + i\epsilon(y-w)^0)^{1-\varepsilon}}$$

For $w^0 > y^0$ this is like a propagator in A^*, for $w^0 > x^0$, like a propagator in A.

This feature leads to pinch surfaces for "cut diagrams", and hence in cross sections.
Then when x and y go to the $+$ direction, w is also pinched in the plus direction.



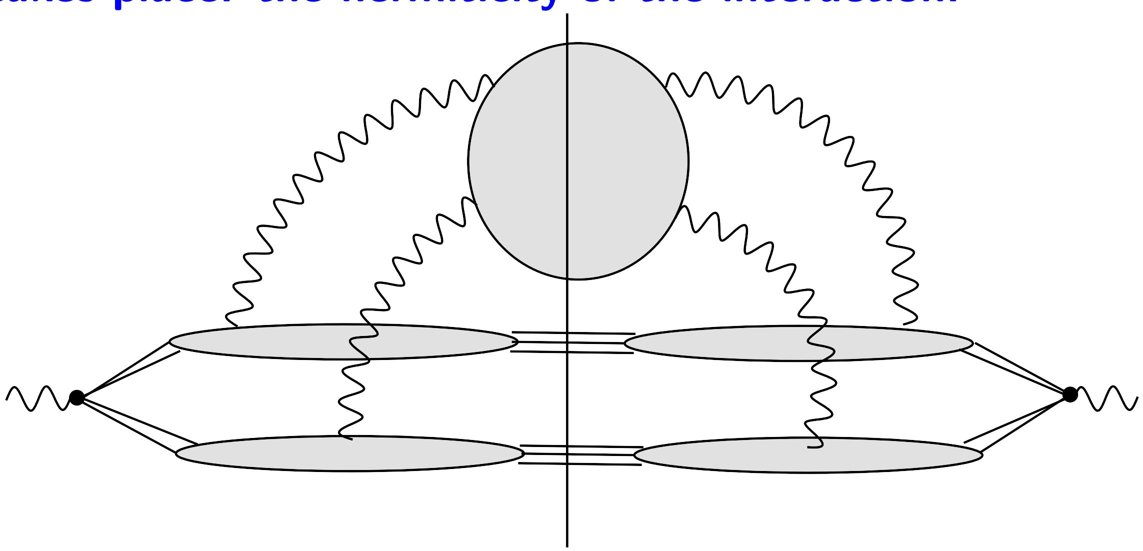
For a pinch, we can assume that $w^0 > x^0$. Then
If $y^0 > w^0$, the "cut" propagator has a $+i\varepsilon$, and produces a pinch just like in the amplitude for $y^- w^+ > 0$. If $x^0 > y^0$, the "cut" propagator has a $-i\varepsilon$, and it still produces a pinch because now $y^- - w^+ < 0$.

All this means that cross sections have the same pinch surfaces as amplitudes, but with energy flow reversed in the complex conjugate. Just like time-ordered perturbation theory, but with the vertices everywhere in coordinate space.

Because vertices of both A and A^* are ordered, the vertex with the largest time is always adjacent to the final state, and connected across the cut.

As for example, in the simplest case of $\mathrm{e}^{+}\mathrm{e}^{-}$ annihilation:

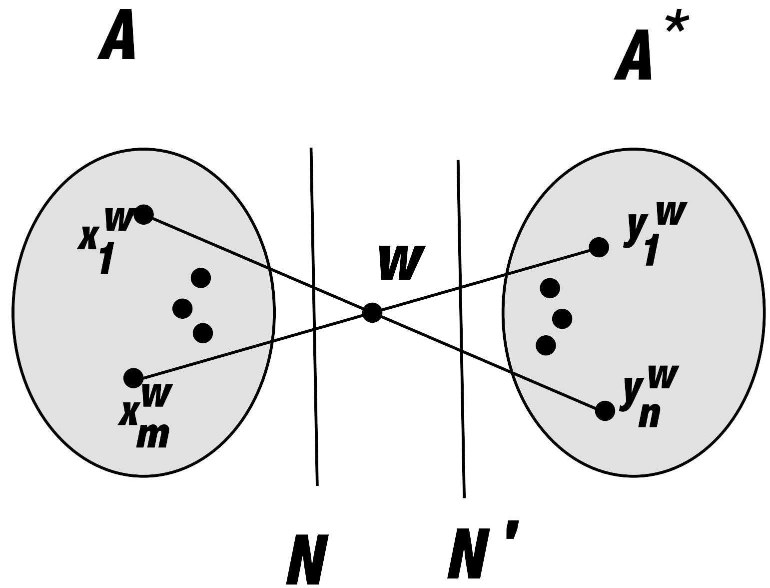
How cancellation takes place: the hermiticity of the interaction.



Cancellation of IR divergences: Let w^0 be the "largest time".
(Veltman 1983 ... R. Akhoury 1992, Laenen, Larsen, Rietkerk 1505.02555)

$$w^0 \;>\; x^0 \quad \text{and} \quad w^0 \;>\; y^0$$

w may be in A or in A^*, and we must sum over both cases: N and N' with vertex at w in A^* and A:



$$\int d^D w \,\Big\{ \prod_j \frac{1}{-(y_j^w - w)^2 + i\epsilon (y_j^w - w)^0}\;[iV(\partial_w)]\;\prod_i \frac{1}{-(w - x_i^w)^2 + i\epsilon}$$
$$+ \prod_j \frac{1}{-(y_j^w - w)^2 - i\epsilon}\;[-iV(\partial_w)]\;\prod_i \frac{1}{-(w - x_i^w)^2 + i\epsilon (w - x_i^w)^0}\Big\}$$

(Typically, $m, n = 1, 2$.)

Vanishes because $w^0 > x^0, y^0$, i.e., is the largest time.

Directions we should go . . . weighted and cut cross sections - all these boil down to top

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle_q = \int d^4x \, \mathrm{e}^{iq\cdot x} \langle 0|O^\dagger(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)O(0)|0\rangle$$

where

$$\mathcal{E}(\vec{n}) = \int_{-\infty}^{\infty} d\tau \lim_{r\to\infty} r^2 n^i T_{0i}(t = \tau + r, r\vec{n})$$

Which can be defined directly in four dimensional space, potentially avoiding an IR-regulated QCD. (. . . Belitsky, Hohenneger, Korchemsky, Sokachev, Zhiboedov, 1311.6800 [exact in N=4 SYM])

Some final thoughts

- **Could we introduce "emergent" degrees of freedom more naturally in coordinate space?**

- **The history of QCD jets and hadronization is there for the reading if only we can learn the language.**