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Four-loop relation between the $\overline{\mathrm{MS}}$ and on-shell quark mass

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Four-loop relation between the $\overline{\mathrm{MS}}$ and on-shell quark mass

I. Motivation
II. Calculation
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Quark mass definitions

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$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G_{\mu\nu}^2 + \sum_q \bar{\psi}_q \left(\not{D} - m_q \right) \psi_q$$

- pole mass
- $\overline{\rm MS}$ mass
- PS mass [Beneke'98]
- 1S mass [Hoang,Smith,Stelzer,Willenbrock'99]
- RS mass [Pineda'01]
- kinetic mass [Bigi,Shifman,Uraltsev,Vainshtein'97]
- ...

Choose quark mass definition in theory calculations
➪ this mass is extracted when comparison with experiment is done

Example: PS mass

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1. defined via relation to poles mass

[Beneke'98]

$$m^{\rm PS}(\mu_f) = m^{\rm OS} - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}\,|<\mu_f} \frac{{\rm d}^3 q}{(2\pi)^3} V(\vec{q}\,)$$

$V(\vec{q}\,)$: static potential
known to 3 loops
[Smirnov,Smirnov,Steinhauser'09; Anzai,Kiyo,Sumino'09]

$$= \mu_f \frac{C_F \alpha_s}{\pi}\left\{1 + \frac{\alpha_s}{4\pi}\left[a_1 + \beta_0\left(2 + \log\frac{\mu^2}{\mu_f^2}\right)\right] + \ldots\right\}$$

➪ $m^{\rm PS} - m^{\rm OS}$ relation known to N^{3}LO
bottom: $\mu_f = 2$ GeV; top: $\mu_f = 20$ GeV;

2. use $m^{\rm OS} - m^{\overline{\rm MS}}$ relation

$$m^{\rm OS} = m^{\overline{\rm MS}}\left(1 + \frac{\alpha_s}{\pi} c_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 c_m^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 c_m^{(3)} + \left(\frac{\alpha_s}{\pi}\right)^4 c_m^{(4)} + \ldots\right)$$

needed to 4 loops

Top and bottom quark mass

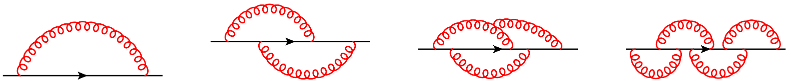
- top quark mass
  - Tevatron/LHC: $m_t^{\mathrm{OS}} = 173.21 \pm 0.51 \pm 0.71$ GeV
    ➪ convert to $\overline{\mathrm{MS}}$ top mass
  - threshold scan at ILC
    ➪ determine in a first step m_t^{PS} or m_t^{1S} or ...
    ➪ convert to $\overline{\mathrm{MS}}$ top mass
- bottom quark mass
  Example: m_b from SRs: $\mathcal{M}_n \equiv \int \mathrm{d}s \frac{R_b(s)}{s^{(n+1)}}$
  - $\overline{\mathrm{MS}}$: m_b from low-moments SRs [Chetyrkin et al.'09, ...]
    $m_b^{\overline{\mathrm{MS}}}(m_b) = 4.163 \pm 0.016$ GeV
  - PS mass: m_b from Υ SRs [Beneke,Maier,Piclum,Rauh'15; ...]
    $m_b^{\mathrm{PS}}(2GeV) = 4.532^{+0.013}_{-0.039}$ GeV
    ➪ convert to $\overline{\mathrm{MS}}$ bottom mass

$Z_m^{\rm OS}$

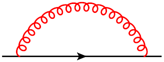
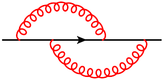
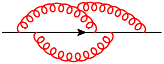
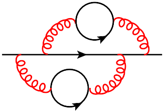
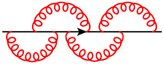
- $$\left.\begin{array}{l} m^{\rm bare} = Z_m^{\overline{\rm MS}} m^{\overline{\rm MS}} \\ m^{\rm bare} = Z_m^{\rm OS} m^{\rm OS} \end{array}\right\} \Longrightarrow m^{\overline{\rm MS}} = m^{\rm OS} \frac{Z_m^{\rm OS}}{Z_m^{\overline{\rm MS}}}$$
- $Z_m^{\overline{\rm MS}}$ known to 4 loops [Chetyrkin'97; Larin,van Ritbergen,Vermaseren'97] (5 loops: [Baikov,Chetyrkin,Kühn'14])
- $Z_m^{\rm OS}$ from

$$S^{-1}(q^2 = m^2) \equiv \not{q} - m + \Sigma(q)\Big|_{q^2=m^2} \stackrel{!}{=} 0$$

$$\Longrightarrow Z_m^{\rm OS} = 1 + \Sigma_V(q^2 = m^2) + \Sigma_S(q^2 = m^2)$$



Z_m^{OS}: known results

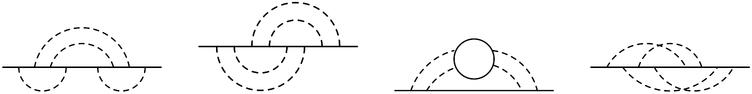
-  [Tarrach'81]
-  [Gray,Broadhurst,Grafe,Schilcher'90]
-  [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]
- n_l^2  [Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]]
- full  [Marquard,Smirnov,Smirnov,Steinhauser'15]

electroweak corrections

[Hempfling,Kniehl'94; Jegerlehner,Kalmykov'03; Faisst,Kühn,Veretin'04; Martin'05; Eiras,Steinhauser'05]

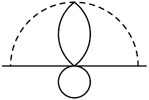
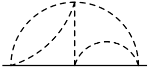
Some technical details

- generation of amplitudes: QGRAF [Nogueira'91]
  manipulation/transformation to FORM: q2e/exp
  [Harlander,Seidelsticker,Steinhauser'97;Seidelsticker'97]
- map to ∼ 100 families (topologies)



- reduce to MIs: FIRE5 [Smirnov'14] and crusher [Marquard,Seidel]
- minimize MIs over all topologies: tsort [Pak'11]
  ➪ 386 4-loop OS MIs
- compute MIs with FIESTA3 [Smirnov'14]

Master integrals

- "simple":
  - factorizable integrals  [Lee,Smirnov'11]
  -  . . . [Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]

Master integrals

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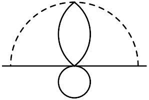
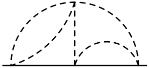
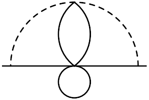
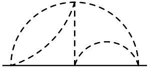
  - factorizable integrals  [Lee,Smirnov'11]
  -  ... [Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]

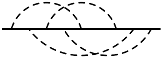
- "medium": apply Mellin-Barnes technique; [MB.m: Czakon]
  (up to) 3-fold MB representation ⇨ $\sim$ 20 digits

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \, \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z)$$

Master integrals

- "simple":
  - factorizable integrals  [Lee,Smirnov'11]
  -  . . . [Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]

- "medium": apply Mellin-Barnes technique; [MB.m: Czakon]
  (up to) 3-fold MB representation ➪ $\sim$ 20 digits

- "hard" ($\approx$ 330):  use `FIESTA3` [Smirnov'14]

```
SDEvaluate[ UF[{k1, k2, k3, k4},
   {-(k1-p1)^2 + MM, -(k1-k3-p1)^2 + MM, -(k1+k2-k3-p1)^2 + MM,
    -(k1+k2-k3-k4-p1)^2 + MM, -(k2-k3-k4-p1)^2 + MM, -(k3+k4+p1)^2 + MM,
    -(k4+p1)^2 + MM, -k1^2, -k2^2, -k3^2, -k4^2},
   {p1^2 -> MM, MM -> 1}], {1,1,1,1,1,1,1,1,1,1,1}, 0]
```

final result: add individual uncertainties in quadrature; $\times 5$ ("5σ")

or: add individual uncertainties linearly

4-loop coefficient

$$m^{\overline{\rm MS}} = m^{\rm OS}\left(1 + \frac{\alpha_s}{\pi} z_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 z_m^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 z_m^{(3)} + \left(\frac{\alpha_s}{\pi}\right)^4 z_m^{(4)} + \ldots\right)$$

charm

$$z_m^{(4)}\Big|_{n_l=3} = -1744.8 \pm 21.5 - 703.48\, l_{\rm OS} - 122.97\, l_{\rm OS}^2 - 14.234\, l_{\rm OS}^3 - 0.75043\, l_{\rm OS}^4$$

bottom

$$z_m^{(4)}\Big|_{n_l=4} = -1267.0 \pm 21.5 - 500.23\, l_{\rm OS} - 83.390\, l_{\rm OS}^2 - 9.9563\, l_{\rm OS}^3 - 0.514033\, l_{\rm OS}^4$$

top

$$z_m^{(4)}\Big|_{n_l=5} = -859.96 \pm 21.5 - 328.94\, l_{\rm OS} - 50.856\, l_{\rm OS}^2 - 6.4922\, l_{\rm OS}^3 - 0.33203\, l_{\rm OS}^4$$

$$l_{\rm OS} = \ln(\mu^2/M^2)$$

- $\ln(\mu^2/M^2)$ known from RGE
- constant term: $< 3\%$ uncertainty

$\overline{\mathrm{MS}}$–OS relation for top and bottom

$$
\begin{aligned}
m_t^{\mathrm{OS}} &= m_t^{\overline{\mathrm{MS}}} \left[1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.49 \pm 0.25)\,\alpha_s^4\right] \\
&= 163.643 + 7.557 + 1.617 + 0.501 + (0.195 \pm 0.005)\ \mathrm{GeV}
\end{aligned}
$$

$\overline{\mathrm{MS}}$–OS relation for top and bottom

$$m_t^{\mathrm{OS}} = m_t^{\overline{\mathrm{MS}}} \left[1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.49 \pm 0.25)\,\alpha_s^4\right]$$
$$= 163.643 + 7.557 + 1.617 + 0.501 + (0.195 \pm 0.005)\ \mathrm{GeV}$$

$$m_b^{\mathrm{OS}} = m_b^{\overline{\mathrm{MS}}} \left[1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 + (12.57 \pm 0.38)\,\alpha_s^4\right]$$
$$= 4.163 + 0.401 + 0.201 + 0.148 + (0.138 \pm 0.004)\ \mathrm{GeV}$$

- top: good/reasonable convergence
- bottom: no convergence

$\overline{\mathrm{MS}}-$ threshold top mass relation

	$m_t^{\overline{\mathrm{MS}}}(m_t^{\overline{\mathrm{MS}}})$			
input	$m^{\mathrm{PS}} =$	$m^{1\mathrm{S}} =$	$m^{\mathrm{RS}} =$	
#loops	171.792	172.227	171.215	
1	165.097	165.045	164.847	
2	163.943	163.861	163.853	1-2 GeV

$\overline{\text{MS}}$ – threshold top mass relation

input #loops	$m^{\text{PS}} =$ 171.792	$m^{\text{1S}} =$ 172.227	$m^{\text{RS}} =$ 171.215	
	$m_t^{\overline{\text{MS}}}(m_t^{\overline{\text{MS}}})$			
1	165.097	165.045	164.847	
2	163.943	163.861	163.853	1-2 GeV
3	163.687	163.651	163.663	$\lesssim$ 250 MeV

$\overline{\mathrm{MS}}-$ threshold top mass relation

	$m_t^{\overline{\mathrm{MS}}}(m_t^{\overline{\mathrm{MS}}})$			
input	$m^{\mathrm{PS}} =$	$m^{1\mathrm{S}} =$	$m^{\mathrm{RS}} =$	
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1	165.097	165.045	164.847	
2	163.943	163.861	163.853	1-2 GeV
3	163.687	163.651	163.663	$\lesssim$ 250 MeV
4	163.643	163.643	163.643	$\lesssim$ 40 MeV

$\overline{\mathrm{MS}}-$ threshold top mass relation

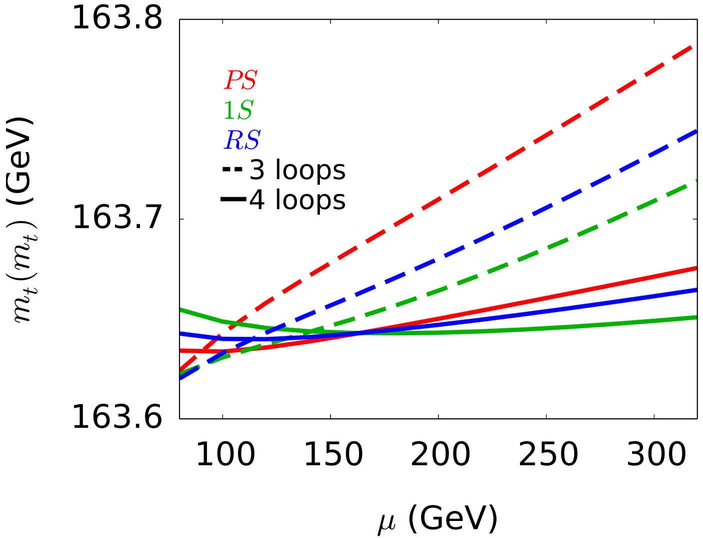
input	$m^{\mathrm{PS}} =$	$m^{\mathrm{1S}} =$	$m^{\mathrm{RS}} =$	
#loops	171.792	172.227	171.215	
	$m_t^{\overline{\mathrm{MS}}}(m_t^{\overline{\mathrm{MS}}})$			
1	165.097	165.045	164.847	
2	163.943	163.861	163.853	1-2 GeV
3	163.687	163.651	163.663	$\lesssim$ 250 MeV
4	163.643	163.643	163.643	$\lesssim$ 40 MeV
4 ($\times$1.03)	163.637	163.637	163.637	

- 3 loops: 200-250 MeV
- 4 loops: $\{44, 8, 20\}$ MeV
- 3% uncertainty $\widehat{=}$ 6 MeV
- $\delta m_t^{\mathrm{ILC}} \lesssim$ 100 MeV

$\Rightarrow$ $\{23, 7, 11\}$ MeV uncertainty in the $m_t^{\overline{\mathrm{MS}}} - m_t^{\mathrm{thr}}$ relation

$\overline{\mathrm{MS}}-$ threshold top mass relation

$$m^{\mathrm{thr}} \longrightarrow m_t^{\overline{\mathrm{MS}}}(\mu) \longrightarrow m_t^{\overline{\mathrm{MS}}}(m_t^{\overline{\mathrm{MS}}})$$



$\overline{\mathrm{MS}}-$ threshold bottom mass relation

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	$m_b^{\overline{\mathrm{MS}}}(m_b^{\overline{\mathrm{MS}}})$			
input	$m^{\mathrm{PS}} =$	$m^{1\mathrm{S}} =$	$m^{\mathrm{RS}} =$	
#loops	4.483	4.670	4.365	
1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV

$\overline{\mathrm{MS}}-$ threshold bottom mass relation

	$m_b^{\overline{\mathrm{MS}}}(m_b^{\overline{\mathrm{MS}}})$			
input	$m^{\mathrm{PS}} =$	$m^{1\mathrm{S}} =$	$m^{\mathrm{RS}} =$	
#loops	4.483	4.670	4.365	
1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV
3	4.161	4.154	4.158	$\lesssim$ 40 MeV

$\overline{\text{MS}}-$ threshold bottom mass relation

	$m_b^{\overline{\text{MS}}}(m_b^{\overline{\text{MS}}})$			
input	$m^{\text{PS}} =$	$m^{1\text{S}} =$	$m^{\text{RS}} =$	
#loops	4.483	4.670	4.365	
1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV
3	4.161	4.154	4.158	$\lesssim$ 40 MeV
4	4.163	4.163	4.163	$\lesssim$ 9 MeV

$\overline{\rm MS}-$ threshold bottom mass relation

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$m_b^{\overline{\rm MS}}(m_b^{\overline{\rm MS}})$

input	$m^{\rm PS} =$	$m^{\rm 1S} =$	$m^{\rm RS} =$	
#loops	4.483	4.670	4.365	
1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV
3	4.161	4.154	4.158	$\lesssim$ 40 MeV
4	4.163	4.163	4.163	$\lesssim$ 9 MeV
4 ($\times$1.03)	4.159	4.159	4.159	

- 3 loops: $\approx$ 40 MeV
- 4 loops: $\{2, 9, 5\}$ MeV
- 3% uncertainty: 4 MeV

$\Rightarrow$ $\{4, 6, 5\}$ MeV uncertainty in the $m_b^{\overline{\rm MS}} - m_b^{\rm thr}$ relation

- N^{3}LO extractions: $\delta m_b \approx 10 - 20$ MeV
- "OS-$\overline{\rm MS}$" – "PS-OS"

$$m_b^{\rm PS} = 4.163 + (0.401 - 0.192) + (0.201 - 0.121) + (0.148 - 0.115) + (0.138 - 0.140)$$

$\overline{\mathrm{MS}}$ – threshold mass formulae

$$\frac{m_t(m_t)}{\text{GeV}} = 163.643 \pm 0.023 + 0.074\Delta_{\alpha_s} - 0.095\Delta_{m_t}^{\mathrm{PS}}$$
$$\frac{m_t(m_t)}{\text{GeV}} = 163.643 \pm 0.007 + 0.069\Delta_{\alpha_s} - 0.096\Delta_{m_t}^{\mathrm{1S}}$$
$$\frac{m_t(m_t)}{\text{GeV}} = 163.643 \pm 0.011 + 0.067\Delta_{\alpha_s} - 0.095\Delta_{m_t}^{\mathrm{RS}}$$

$$\frac{m_b(m_b)}{\text{GeV}} = 4.163 \pm 0.004 + 0.007\Delta_{\alpha_s} - 0.018\Delta_{m_b}^{\mathrm{PS}}$$
$$\frac{m_b(m_b)}{\text{GeV}} = 4.163 \pm 0.006 + 0.008\Delta_{\alpha_s} - 0.019\Delta_{m_b}^{\mathrm{1S}}$$
$$\frac{m_b(m_b)}{\text{GeV}} = 4.163 \pm 0.005 + 0.004\Delta_{\alpha_s} - 0.018\Delta_{m_b}^{\mathrm{RS}}$$

$\Delta_{\alpha_s} = (0.1185 - \alpha_s(M_Z))/0.001$, $\Delta_{m_t}^{\mathrm{PS}} = (171.792\text{ GeV} - m_t^{\mathrm{PS}})/0.1$

...

$\overline{\mathrm{MS}}$–OS relation: compare to approximations

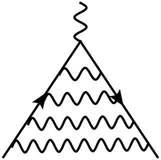
	$c_m^{(4)}(\mu = m(m))$			
	$n_l = 3$	$n_l = 4$	$n_l = 5$	Method
[Beneke,Braun'94]	1668	1324	1031	large-β_0
[Chetyrkin,Kniehl,Sirlin'97]	1571.4	1107.8	727.0	PMS, FAC, ...
[Kataev,Kim'10]	1281	986	719	"π^2"
[Ayala,Cvetic'12]	1785.9	1316.4	920.1	ren.cancel
[Sumino'13]	1668 ± 167	1258^{+26}_{-66}	897^{+31}_{-175}	ren.cancel
[Ayala,Cvetic,Pineda'14]	1772 ± 82	1324 ± 82	945 ± 92	ren.cancel
[Marquard,Smirnov, Smirnov,Steinhauser'15]	1691.2 ± 21.5	1224.0 ± 21.5	827.37 ± 21.5	

$$m^{\mathrm{OS}} = m^{\overline{\mathrm{MS}}} \left(1 + \frac{\alpha_s}{\pi} c_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 c_m^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 c_m^{(3)} + \left(\frac{\alpha_s}{\pi}\right)^4 c_m^{(4)} + \ldots\right)$$

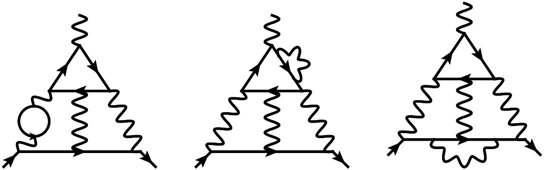
Further application:
4-loop QED corrections to g-2

[Aoyama,Hayakawa,Kinoshita,Nio'07'08'12]

- $(g-2)_e$, photonic part of $(g-2)_\mu$



- $(g-2)_\mu$: most important: light-by-light-type with e^- loops



additional complication: $m_e \neq 0$ ➪ asymptotic expansion in m_e/m_μ

Conclusions

- Systematic study of 4-loop OS integrals
- 4-loop $\overline{\mathrm{MS}}$-OS relation for heavy quarks
- Precise $m^{\overline{\mathrm{MS}}}$-$m^{\mathrm{thr}}$ relation to N^{3}LO