Five-loop massive tadpoles

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recent work with Thomas Luthe

and earlier work with:
J. Möller, C. Studerus

Radcor15, UCLA, 16 Jun 2015
Motivation

• pressure of hot QCD
  ▶ phenomenology needs physical NLO
  ▶ hardest building block: 5-loop tadpoles(m,0) at \( d = 3 - 2\epsilon \) [Braaten/Nieto 96; KLRS 04]

• QCD beta function and anomalous dimensions
  ▶ known since \( \sim 18 \) years at 4-loop accuracy [vanRitbergen/Vermaseren/Larin 97]
  ▶ 5-loop needed e.g. for improved \( \rho \)-parameter, decoupling equation, ...
  ▶ partial results appear since \( \sim 5 \) years
    mainly from Karlsruhe (\( \beta^{QED} \), \( \gamma_m \), ...)
    [Baikov/Chetyrkin/Kühn/Rittinger 08-13]

• moments
  ▶ many problems allow for asymptotic expansions
  ▶ mapping on tadpoles, often for price of many dots

• in this talk: focus on master integrals
  ▶ basic building block for 5-loop problems
  ▶ interested in methods that allow to choose \( d \) in the end
  ▶ main progress via refinement of Laporta approach
  ▶ (problem-specific) reduction not covered here
    \[ \Rightarrow \text{see talk of T.Luthe} \]
    \[ \Rightarrow \text{see other talks} \]
Classification

- consider fully massive 5-loop tadpoles
  - Euclidean space-time
  - same mass in all propagators $\Rightarrow 1/(q_i^2 + 1)$

- the 5-loop integral family needs 15 propagators / lines
  - $q_i \in \{k_1, k_2, k_3, k_4, k_5, k_{13}, k_{14}, k_{15}, k_{23}, k_{24}, k_{25}, k_{35}, k_{45}, k_{124}, k_{34}\}$
    where $k_{a...bc} = k_a + \ldots + k_b - k_c$
  - trivalent graphs have 12 lines:

- classification: label sectors by binary rep
  - identify unique graphs
  - find all isometries and corresponding momentum shifts
  - choose largest representative from each class

- normalization: divide out $[1\text{-loop tadpole}]^{\#loops}$
  - recall that in 4d, $[1\text{-loop tadpole}] \sim 1/\epsilon$
Classification: non-trivial shift relations

- suppose you were given a 5-loop massive tadpole

\[
\int_{k_{1..5}}^{(d)} \frac{1}{k_4^2 + 1} \frac{1}{(k_1 - k_3)^2 + 1} \frac{1}{(k_1 - k_5)^2 + 1} \frac{1}{(k_2 - k_3)^2 + 1} \frac{1}{(k_2 - k_5)^2 + 1} \frac{1}{(k_3 - k_5)^2 + 1} \frac{1}{(k_1 + k_2 - k_4)^2 + 1}
\]

shift \( k_i^\mu \rightarrow k_i'^\mu = M_{ij} k_j^\mu \) with \( M = \frac{1}{2} \left( \begin{array}{cccccc}
0 & 1 & 1 & -1 & 1 \\
0 & 1 & -1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 2 & 0 & 0 & 0 \\
-2 & 1 & 1 & 1 & 1
\end{array} \right) \)

and \( |\det M| = \frac{1}{2} \) then gives

\[
= 2^{-d} \int_{k_{1..5}}^{(d)} \frac{1}{k_2^2 + 1} \frac{1}{k_4^2 + 1} \frac{1}{(k_1 - k_4)^2 + 1} \frac{1}{k_3^2 + 1} \frac{1}{(k_1 - k_3)^2 + 1} \frac{1}{k_1^2 + 1} \frac{1}{k_5^2 + 1}
\]

\[
= 2^{-d} \times \text{diagram}
\]

- does this (evil) momentum-labelling occur in practice?
Classification: 5-loop

• some numerology
  ▶ for specific momentum list
  ▶ combinatorics wins!

• \( L(L+1)/2 + LE = 15 \) scalar products

• \( 2^{15} = 32768 \) possible sectors

• 121 + 13332 do not correspond to a Feynman graph

• 1941 + 3361 zero-sectors

• 13946 shifts

• 19 + 48 unique sectors
  (16 with 1-loop factors not shown)
Classification: 6-loop

- 21 sc prod
- $2^{21} = 2.1$M sectors
- 1.3M no graph
- 178K zeros
- 569K shifts
- 487 unique
  (show 3-conn. cubic graphs)
5-loop Sectors

- arrive at 48 unique 5-loop sectors (+19 factorized ones not shown)

$t = 6$

$t = 7$

$t = 8$

$t = 9$

$t = 10$

$t = 11$

$t = 12$

▷ recall that at 1/2/3/4-loop there were 1/1/3/10 unique sectors (plus 0/1/2/6 fact)
5-loop Masters

- a (small) IBP reduction reveals that some sectors contain multiple master integrals
  - need in addition 62 (+3 factorized ones) masters with 'dots'. some examples:
    - recall that at 1/2/3/4-loop there were 0/0/0/3 masters with 'dots'

- how to evaluate these 48+62 (+19+3) zero-scale master integrals?
  - various methods, e.g.
    - explicit integration in x-space
    - differential eqs (in mass ratio); solve iteratively with HPLs
    - explicit solution of low-order difference equations: $pF_{P-1}$ etc.
    - numerical solution of difference equations via factorial series

- Mathematical structure
  - interested in the coefficients of an $\epsilon$ expansion
  - in many cases, these are from a generic class of functions/numbers
    - e.g. harmonic polylogarithms HPL(x)  
    - e.g. harmonic sums $S(N)$
    - relation: $H_m(1) \rightarrow S_m(\infty)$
  - if solution numerical: use some PSLQ
Evaluation: differential equations

• perform IBP reduction with two masses: \( M, m \)
  ▶ get differential eqn in \( z = M/m \)
  ▶ use boundary values at \( z = 0 \) (\( z = 1 \))
  ▶ use symmetry relations like \( z \leftrightarrow 1/z \)
  ▶ typically, want the integral at \( z = 1 \) (\( z = 0 \))

• simple 3-loop example (basketball type)

\[
B_{24}(z) \equiv \frac{1}{j^3} \int_{p_{1..3}}^{(d)} \frac{1}{p_1^2 + z^2} \frac{1}{p_2^2 + z^2} \frac{1}{p_3^2 + 1} \frac{1}{(p_1 + p_2 + p_3)^2 + 1}
\]

\[
B_{24}(z) = x^{3d-8} B_{24}(1/z), \quad B_{24}(0) = 2^{d-3} \frac{\Gamma\left(\frac{8-3d}{2}\right)\Gamma\left(\frac{3-d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{7-2d}{2}\right)\Gamma\left(\frac{2-d}{2}\right)}
\]

satisfies \( \{ z(1 - z^2)\partial_z^2 - 2(1 - 2z^2)(d - 3)\partial_z - z(d - 3)(3d - 8) \} B_{24}(z) = (d - 2)^2 z^{d-3} (z^{d-2} - 1) \)

• solution standard, via variation of constants, in terms of HPL(\( z \));
  set \( z = 1 \) and use algebra of HPL(1) resp. \( S(\infty) \)

\[
B_4 = -2 + \ldots
\]

\[
+ e^7 \cdot 4 \cdot (1141/24 - 112/3*3) \quad + \ldots
\]

\[
+ e^7 \cdot (418903/192 + 2278/45*pi^4 + 32/5*pi^6 - 3840*s6 - \ldots) \quad + \ldots
\]

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Evaluation: difference equations

- perform IBP reduction with symbolic power $x$ on one line

- derive difference equation for generalized master $I(x) \equiv \int \frac{1}{D_1 D_2 \cdots D_N} \sum_{j=0}^{R} p_j(x) I(x + j) = F(x)$

- typically, want $I(1)$; solve the difference equation
  - explicitly (if 1st order)
  - numerically (very general setup) [Laporta 00] [see talk by T.Luthe]

- solve via factorial series $I(x) = I_0(x) + \sum_{j=1}^{R} I_j(x)$, where
  \[ I_j(x) = \mu_j x^j \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x + 1)}{\Gamma(x + 1 + s - K_j)} \]

- need boundary condition for fixing, say, $a_j(0)$: use decoupling at large $x$
  - $I(x) = \int k_1 g(k_1)/(k_1^2 + 1)^x \Rightarrow I(x) \sim (1)^x x^{-d/2} g(0)$
Choice of basis of transcendentals

• to absorb single powers of $\pi$ as well as powers of $\ln 3$, def

$$h_n \equiv \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)}{\Gamma(k + 1)\Gamma(1/2)} \frac{(3/4)^k}{(2k + 1)^n}$$

$$H_n \equiv h_n + h_1 \text{Coefficient} \left[ 1 - \frac{3^{\epsilon/2}\Gamma(1 - \epsilon)}{\Gamma^2(1 - \epsilon/2)} + \mathcal{O}(\epsilon^n), \epsilon, n - 1 \right]$$

$$H_1 = h_1 = \frac{2\pi}{3\sqrt{3}} , \quad H_2 = h_2 - \frac{1}{2}h_1 \ln 3 , \quad etc$$

$\triangleright \quad I_{7.1.1} = +\left(\frac{3}{2}\right)\epsilon^0 + \left(\frac{3}{2}\right)\epsilon^1 + (9H_2 - 3)\epsilon^2 + (9H_2 - 18H_3 - 6)\epsilon^3 + (18H_2 - 18H_3 + 36H_4 - 12)\epsilon^4 + \ldots$

• to absorb powers of $\ln 2$, def as elements of the MZV basis

$$A_n \equiv Li_n\left(\frac{1}{2}\right) + (-1)^n \frac{\ln^n 2}{n!} \left(1 - n(n - 1) \frac{\zeta_2}{\ln^2 2}\right)$$

$\triangleright \quad I_{63.1.1} = +(0)\epsilon^0 + (0)\epsilon^1 + (-2\zeta_3)\epsilon^2 + (-16A_4 + 27H_2^2 + \frac{34\zeta_2^2}{5})\epsilon^3 + \ldots$
Sample results (4d)

\[
\begin{align*}
I_{28686.1.1} &= +(-3)\epsilon^0 + \left(-\frac{3}{2}\right)\epsilon^1 + \left(\frac{13}{24}\right)\epsilon^2 + \left(-\frac{1267}{1440}\right)\epsilon^3 + \left(-\frac{4193}{3456}\right)\epsilon^4 + \\
&\quad + 135.95072868792871461956492733702218574897992953584\epsilon^5 + \ldots \\
I_{28686.1.3} &= +(0)\epsilon^0 + \left(\frac{3}{2}\right)\epsilon^1 + \left(-\frac{1}{2}\right)\epsilon^2 + \left(-\frac{443}{360}\right)\epsilon^3 + \left(\frac{95}{216}\right)\epsilon^4 + \\
&\quad - 38.292059175062436961881799538284449799148385376441\epsilon^5 + \ldots \\
I_{30862.1.1} &= +\left(-\frac{3}{5}\right)\epsilon^0 + \left(-\frac{27}{10}\right)\epsilon^1 + \left(-\frac{4\zeta_3}{5} - \frac{421}{60}\right)\epsilon^2 + \left(-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24}\right)\epsilon^3 + \left(\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48}\right)\epsilon^4 + \\
&\quad + 1143.1838307558764599466030303839590323268318605888\epsilon^5 + \ldots \\
I_{30862.1.2} &= +\left(\frac{1}{5}\right)\epsilon^0 + \left(\frac{11}{30}\right)\epsilon^1 + \left(-\frac{1}{30}\right)\epsilon^2 + \left(-\frac{12\zeta_3}{5} - 9\right)\epsilon^3 + \left(-\frac{36\zeta_2^2}{25} + \frac{548\zeta_3}{15} - \frac{1229}{15}\right)\epsilon^4 + \\
&\quad - 102.42854342605587086319606311891941160276036031953\epsilon^5 + \ldots \\
I_{30231.1.1} &= +(0)\epsilon^0 + \left(0\right)\epsilon^1 + \left(\frac{3\zeta_3}{5}\right)\epsilon^2 + \left(\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5\right)\epsilon^3 + \left(-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5}\right)\epsilon^4 + \\
&\quad - 531.32391547725635267943444561495368318398901378435\epsilon^5 + \ldots \\
I_{31420.1.1} &= +(0)\epsilon^0 + \left(0\right)\epsilon^1 + \left(0\right)\epsilon^2 + \left(0\right)\epsilon^3 + \left(-\frac{36\zeta_2^2}{5}\right)\epsilon^4 + \\
&\quad + 167.815353059184740619621206011112466233675898298296\epsilon^5 + \ldots
\end{align*}
\]
Invitation

to UTFSM Valparaíso, Chile

13-15 Jan 2016: 4th Chilean HEP (Summer!) School

18-22 Jan 2016: ⇒ 17th ACAT workshop ⇐

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