Five-loop massive tadpoles

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recent work with Thomas Luthe

and earlier work with:
J. Möller, C. Studerus

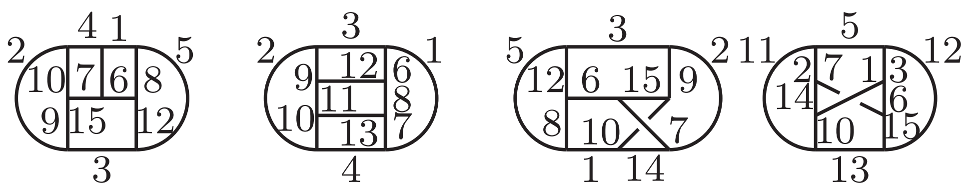
Radcor15, UCLA, 16 Jun 2015

Motivation

- pressure of hot QCD
  - ▷ phenomenology needs physical NLO [Braaten/Nieto 96; KLRS 04]
  - ▷ hardest building block: 5-loop tadpoles(m,0) at $d = 3 - 2\epsilon$
- QCD beta function and anomalous dimensions
  - ▷ known since ∼18 years at 4-loop accuracy [vanRitbergen/Vermaseren/Larin 97]
  - ▷ 5-loop needed e.g. for improved ρ-parameter, decoupling equation, ... [4loop: 06]
  - ▷ partial results appear since ∼5 years
    mainly from Karlsruhe (β^{QED}, γ_m, ...) [Baikov/Chetyrkin/Kühn/Rittinger 08-13]
- moments
  - ▷ many problems allow for asymptotic expansions
  - ▷ mapping on tadpoles, often for price of many dots
- in this talk: focus on master integrals
  - ▷ basic building block for 5-loop problems
  - ▷ interested in methods that allow to choose d in the end
  - ▷ main progress via refinement of Laporta approach [⇒ see talk of T.Luthe]
  - ▷ (problem-specific) reduction not covered here [⇒ see other talks]

Classification

- consider fully massive 5-loop tadpoles
  - ▷ Euclidean space-time
  - ▷ same mass in all propagators $\Rightarrow 1/(q_i^2+1)$
- the 5-loop integral family needs 15 propagators / lines
  - ▷ $q_i \in \{k_1, k_2, k_3, k_4, k_5, k_{13}, k_{14}, k_{15}, k_{23}, k_{24}, k_{25}, k_{35}, k_{45}, k_{124}, k_{34}\}$
    where $k_{a\ldots bc} = k_a + \ldots + k_b - k_c$
  - ▷ trivalent graphs have 12 lines:



- classification: label sectors by binary rep
  - ▷ identify unique graphs
  - ▷ find all isometries and corresponding momentum shifts
  - ▷ choose largest representative from each class
- normalization: divide out [1-loop tadpole]$^{\#loops}$
  - ▷ recall that in 4d, [1-loop tadpole]$\sim 1/\epsilon$

Classification: non-trivial shift relations

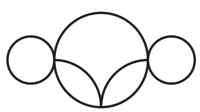
- suppose you were given a 5-loop massive tadpole [ID 2778 of our list above]

$$\int_{k_{1..5}}^{(d)} \frac{1}{k_4^2+1}\,\frac{1}{(k_1-k_3)^2+1}\,\frac{1}{(k_1-k_5)^2+1}\,\frac{1}{(k_2-k_3)^2+1}\,\frac{1}{(k_2-k_5)^2+1}\,\frac{1}{(k_3-k_5)^2+1}\,\frac{1}{(k_1+k_2-k_4)^2+1}$$

shift $\quad k_i^\mu \to k'^\mu_i = M_{ij}k_j^\mu \quad$ with $\quad M = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ -2 & 1 & 1 & 1 & 1 \end{pmatrix}$

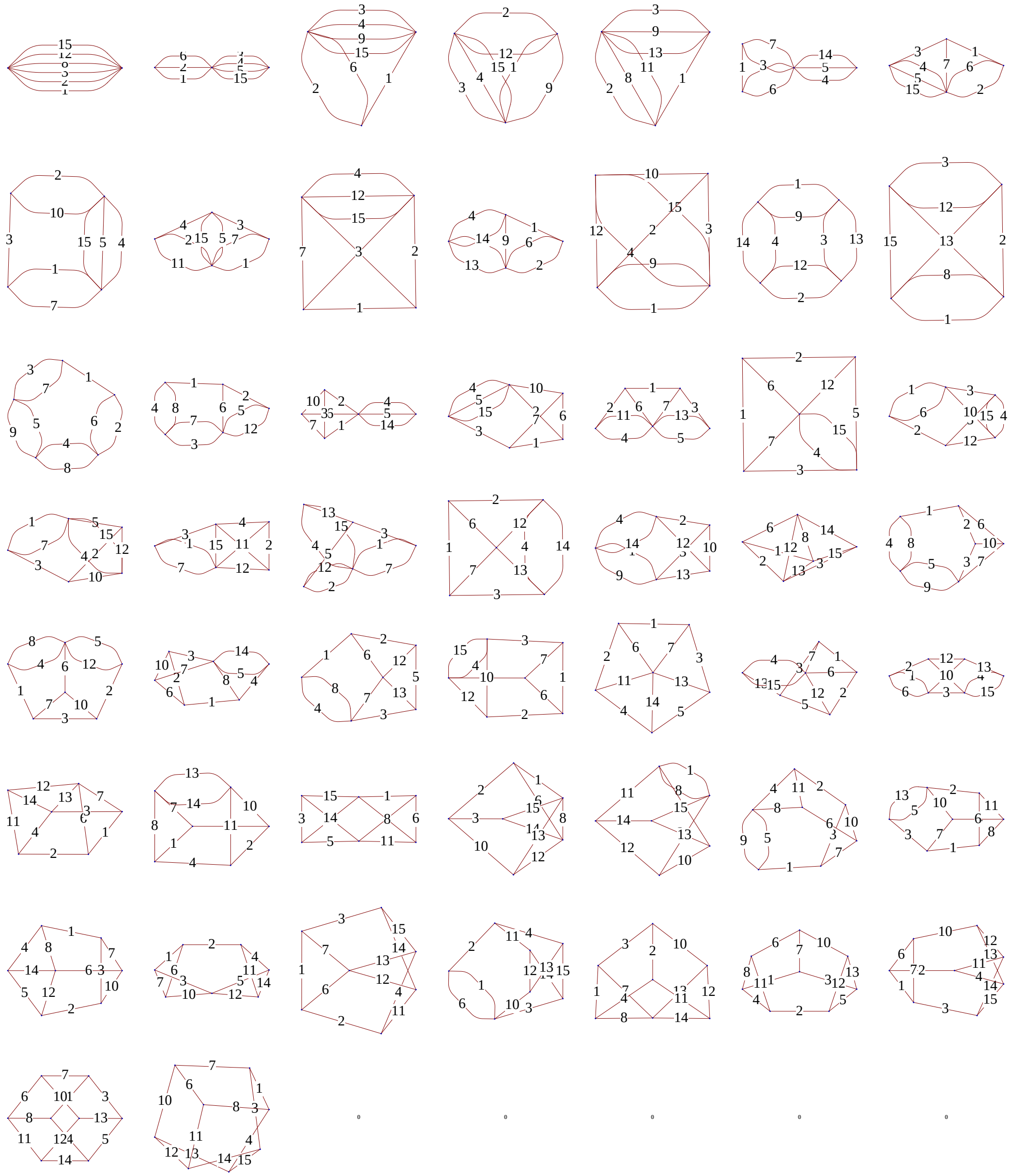
and $\quad |\,\mathbf{det}\,M| = \frac{1}{2} \quad$ then gives

$$= 2^{-d}\int_{k_{1..5}}^{(d)} \frac{1}{k_2^2+1}\,\frac{1}{k_4^2+1}\,\frac{1}{(k_1-k_4)^2+1}\,\frac{1}{k_3^2+1}\,\frac{1}{(k_1-k_3)^2+1}\,\frac{1}{k_1^2+1}\,\frac{1}{k_5^2+1}$$

$= 2^{-d}\times$  [ID 32512 of our list above]

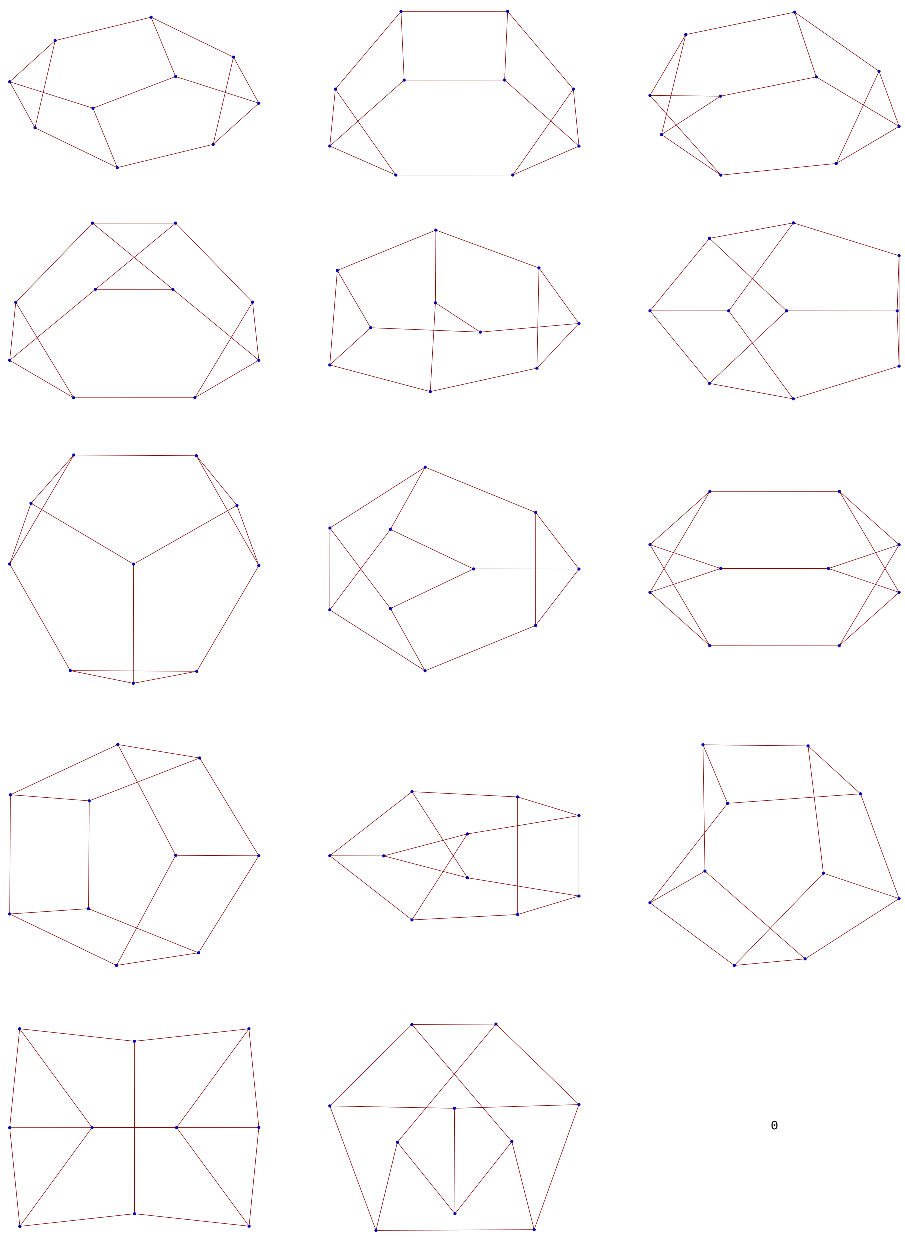
- does this (evil) momentum-labelling occur in practice?

Classification: 5-loop



- some numerology
  - ▷ for specific momentum list
  - ▷ combinatorics wins!
- $L(L+1)/2 + LE = 15$ scalar products
- $2^{15} = 32768$ possible sectors
- 121 + 13332 do not correspond to a Feynman graph
- 1941 + 3361 zero-sectors
- 13946 shifts
- 19 + 48 unique sectors
  (16 with 1-loop factors not shown)

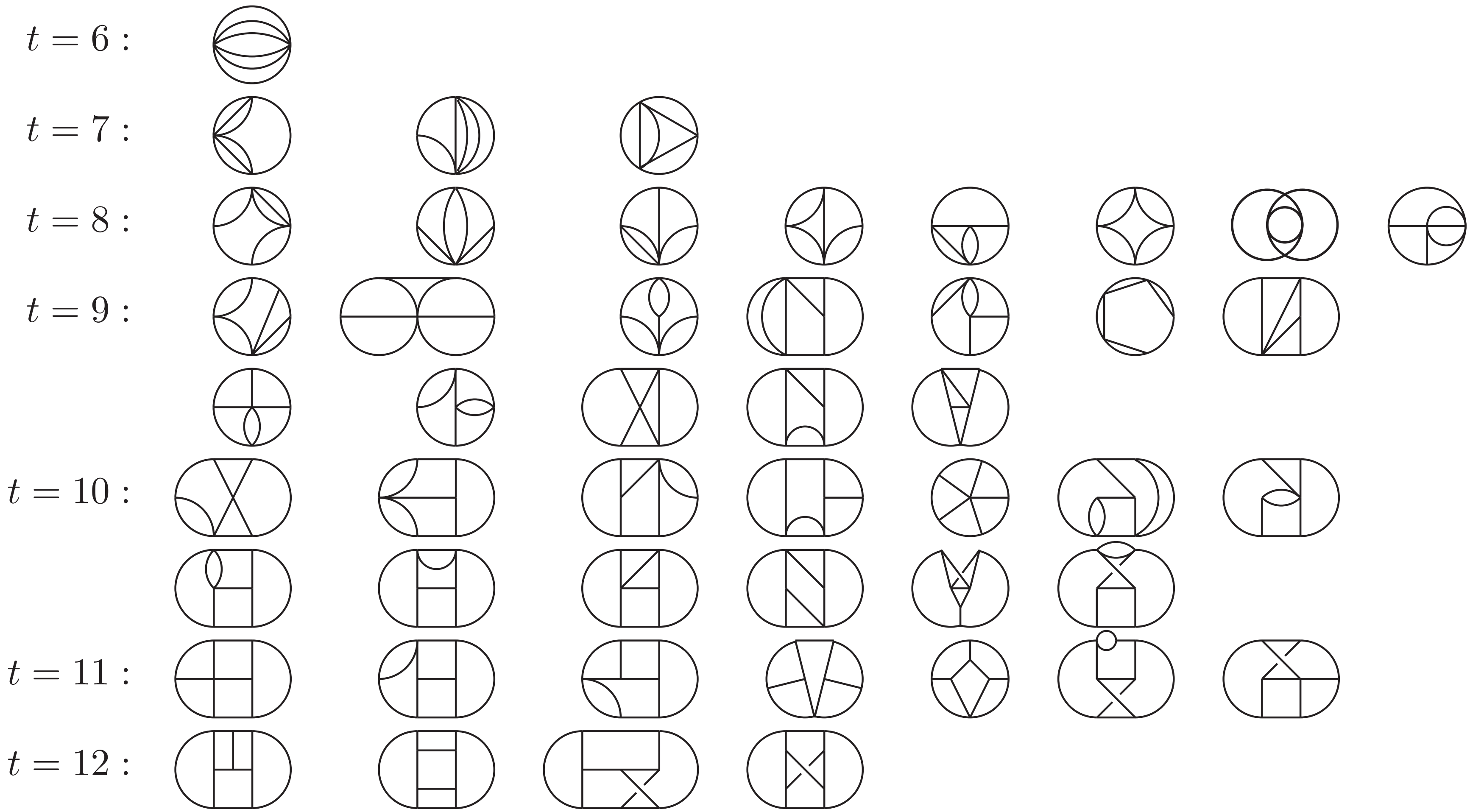
Classification: 6-loop



- 21 sc prod
- 2^{21} = 2.1M sectors
- 1.3M no graph
- 178K zeros
- 569K shifts
- 487 unique
  (show 3-conn. cubic graphs)

5-loop Sectors

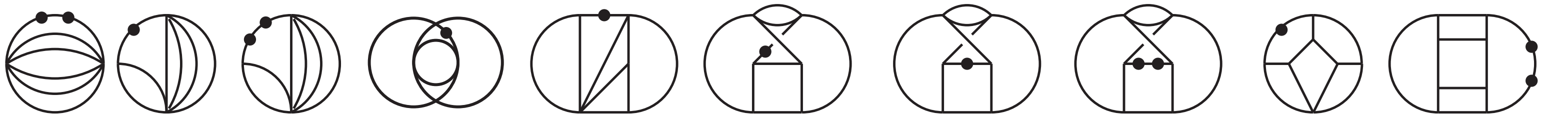
- arrive at 48 unique 5-loop sectors (+19 factorized ones not shown)



▷ recall that at 1/2/3/4-loop there were 1/1/3/10 unique sectors (plus 0/1/2/6 fact)

5-loop Masters

- a (small) IBP reduction reveals that some sectors contain multiple master integrals
  - ▷ need in addition 62 (+3 factorized ones) masters with 'dots'. some examples:



  - ▷ recall that at 1/2/3/4-loop there were 0/0/0/3 masters with 'dots'

- how to evaluate these 48+62 (+19+3) zero-scale master integrals?
  various methods, e.g.
  - ▷ explicit integration in x-space
  - ▷ differential eqs (in mass ratio); solve iteratively with HPLs
  - ▷ explicit solution of low-order difference equations: ${}_PF_{P-1}$ etc.
  - ▷ numerical solution of difference equations via factorial series [Laporta 00]

- Mathematical structure
  - ▷ interested in the coefficients of an ϵ expansion
  - ▷ in many cases, these are from a generic class of functions/numbers
  - ▷ e.g. harmonic polylogarithms HPL(x) [Remiddi/Vermaseren 00]
  - ▷ e.g. harmonic sums $S(N)$ [Vermaseren 98]
  - ▷ relation: $H_{\bar{m}}(1) \to S_{\bar{m}}(\infty)$
  - ▷ if solution numerical: use some PSLQ

Evaluation: differential equations

- perform IBP reduction with two masses: M, m
  - ▷ get differential eqn in $z = M/m$
  - ▷ use boundary values at $z = 0$ ($z = 1$)
  - ▷ use symmetry relations like $z \leftrightarrow 1/z$
  - ▷ typically, want the integral at $z = 1$ ($z = 0$)

- simple 3-loop example (basketball type)

$$B_{24}(z) \equiv \frac{1}{J^3}\int_{p_{1..3}}^{(d)} \frac{1}{p_1^2+z^2}\,\frac{1}{p_2^2+z^2}\,\frac{1}{p_3^2+1}\,\frac{1}{(p_1+p_2+p_3)^2+1}$$

$$B_{24}(z) = x^{3d-8}\,B_{24}(1/z) \quad , \quad B_{24}(0) = 2^{d-3}\,\frac{\Gamma(\frac{8-3d}{2})\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{7-2d}{2})\Gamma(\frac{2-d}{2})}$$

satisfies $\Big\{z(1-z^2)\partial_z{}^2 - 2(1-2z^2)(d-3)\partial_z - z(d-3)(3d-8)\Big\}\,B_{24}(z) \;=\; (d-2)^2\,z^{d-3}\left(z^{d-2}-1\right)$

- solution standard, via variation of constants, in terms of HPL(z);
  set $z = 1$ and use algebra of HPL(1) resp. $S(\infty)$

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B4 = - 2 + ...

     + ep^4 * ( 1141/24 - 112/3*z3 ) + ...

     + ep^7 * ( 418903/192 + 2278/45*pi^4 + 32/5*pi^6 - 3840*s6 - ...) + ...
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Evaluation: difference equations

- perform IBP reduction with symbolic power x on one line
- derive difference equation for generalized master $I(x) \equiv \int \frac{1}{D_1^x D_2 \dots D_N}$

$$\sum_{j=0}^{R} p_j(x) I(x+j) \quad = \quad F(x)$$

- typically, want $I(1)$; solve the difference equation
  - ▷ explicitly (if 1st order)
  - ▷ numerically (very general setup) [Laporta 00] [see talk by T.Luthe]
- solve via factorial series $I(x) = I_0(x) + \sum_{j=1}^{R} I_j(x)$, where

$$I_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$$

- need boundary condition for fixing, say, $a_j(0)$: use decoupling at large x
  - ▷ $I(x) = \int_{k_1} g(k_1)/(k_1^2+1)^x \;\Rightarrow\; I(x) \sim (1)^x x^{-d/2} g(0)$

Choice of basis of transcendentals

- to absorb single powers of π as well as powers of $\ln 3$, def

$$h_n \equiv \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(k+1)\Gamma(1/2)} \frac{(3/4)^k}{(2k+1)^n}$$

$$H_n \equiv h_n + h_1 \,\text{Coefficient}\left[1 - \frac{3^{\epsilon/2}\Gamma(1-\epsilon)}{\Gamma^2(1-\epsilon/2)} + \mathcal{O}(\epsilon^n), \epsilon, n-1\right]$$

$$H_1 = h_1 = \frac{2\pi}{3\sqrt{3}}\,, \quad H_2 = h_2 - \frac{1}{2} h_1 \ln 3\,, \quad etc$$

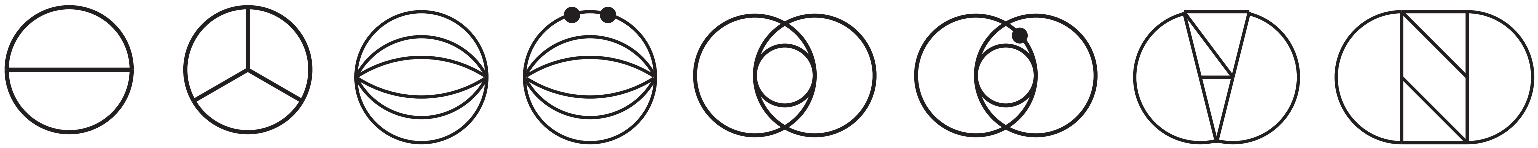
▷ $I_{7.1.1} = +(-\frac{3}{2})\,\epsilon^0 + (-\frac{3}{2})\,\epsilon^1 + (9H_2 - 3)\,\epsilon^2 + (9H_2 - 18H_3 - 6)\,\epsilon^3 + (18H_2 - 18H_3 + 36H_4 - 12)\,\epsilon^4 + \ldots$

- to absorb powers of $\ln 2$, def as elements of the MZV basis

$$A_n \equiv Li_n(\frac{1}{2}) + (-1)^n \frac{\ln^n 2}{n!}\left(1 - \frac{n(n-1)}{2}\frac{\zeta_2}{\ln^2 2}\right)$$

▷ $I_{63.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (-2\zeta_3)\,\epsilon^2 + (-16A_4 + 27H_2^2 + \frac{34\zeta_2^2}{5})\,\epsilon^3 + \ldots$

Sample results (4d)



$$I_{28686.1.1} = +(-3)\,\epsilon^0 + (-\frac{3}{2})\,\epsilon^1 + (\frac{13}{24})\,\epsilon^2 + (-\frac{1267}{1440})\,\epsilon^3 + (-\frac{4193}{3456})\,\epsilon^4 + \\ +135.95072868792871461956492733702218574897992953584\,\epsilon^5 + \ldots$$

$$I_{28686.1.3} = +(0)\,\epsilon^0 + (\frac{3}{2})\,\epsilon^1 + (-\frac{1}{2})\,\epsilon^2 + (-\frac{443}{360})\,\epsilon^3 + (\frac{95}{216})\,\epsilon^4 + \\ -38.292059175062436961881799538284449799148385376441\,\epsilon^5 + \ldots$$

$$I_{30862.1.1} = +(-\frac{3}{5})\,\epsilon^0 + (-\frac{27}{10})\,\epsilon^1 + (-\frac{4\zeta_3}{5} - \frac{421}{60})\,\epsilon^2 + (-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24})\,\epsilon^3 + (\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48})\,\epsilon^4 + \\ +1143.1838307558764599466030303839590323268318605888\,\epsilon^5 + \ldots$$

$$I_{30862.1.2} = +(\frac{1}{5})\,\epsilon^0 + (\frac{11}{30})\,\epsilon^1 + (-\frac{1}{30})\,\epsilon^2 + (-\frac{12\zeta_3}{5} - 9)\,\epsilon^3 + (-\frac{36\zeta_2^2}{25} + \frac{548\zeta_3}{15} - \frac{1229}{15})\,\epsilon^4 + \\ -102.42854342605587086319606311891941160276036031953\,\epsilon^5 + \ldots$$

$$I_{30231.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (\frac{3\zeta_3}{5})\,\epsilon^2 + (\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5)\,\epsilon^3 + (-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_3^2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5})\,\epsilon^4 \\ -531.32391547725635267943444561495368318398901378435\,\epsilon^5 + \ldots$$

$$I_{31420.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-\frac{36\zeta_3^2}{5})\,\epsilon^4 + \\ +167.81535305918474061962120601112466233675898298296\,\epsilon^5 + \ldots$$

Invitation

to UTFSM Valparaíso, Chile

13-15 Jan 2016: 4th Chilean HEP (Summer!) School [https://indico.cern.ch/event/394935]

18-22 Jan 2016: ⇒ 17th ACAT workshop ⇐ [https://indico.cern.ch/event/397113]