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Calculation of 3-Loop massive ladder and V-Diagrams with difference-ring techniques

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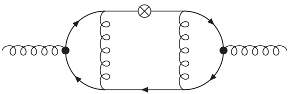
joint with A. Behring, J. Blümlein, A. De Freitas (DESY)
and J. Ablinger (RISC) and A. von Manteuffel (U. Mainz)

Der Wissenschaftsfonds

Goal: Calculate the 3-loop massive ladder and V-diagrams



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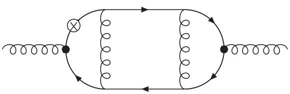

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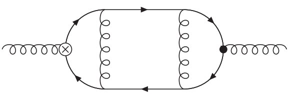

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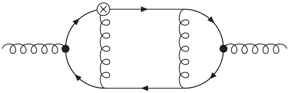

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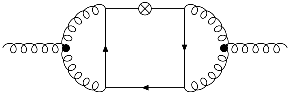

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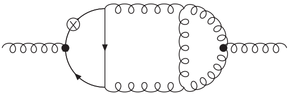

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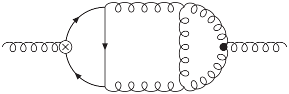

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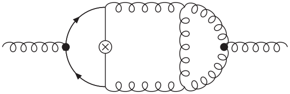

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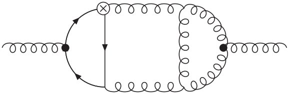

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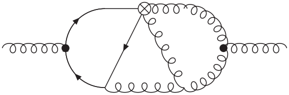

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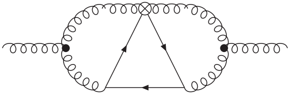

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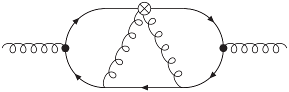

12

All diagrams are produced with axodraw (J. Vermaseren)

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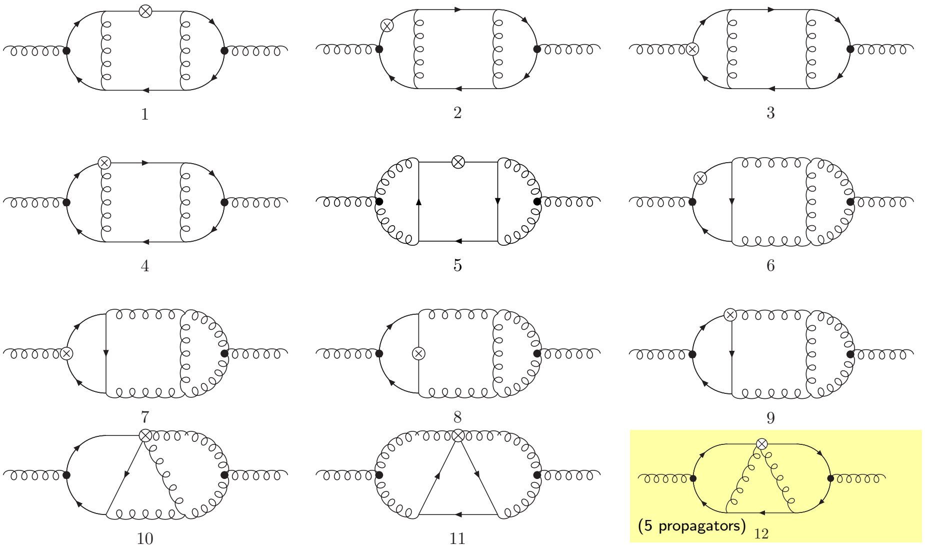
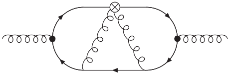


All diagrams are produced with axodraw (J. Vermaseren)

Calculate the first coefficients in the ε-expansion

$D_{12}(N) =$ 

12

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^{0} + \dots$$

Consider the power series of $D_{12}(N)$:

$$D_{12}(N) \longrightarrow \hat{D}_{12}(x) = \sum_{N=0}^{\infty} D_{12}(N) x^N$$

(holonomic closure properties)

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$$\sum_{N=0}^{\infty} D_{12}(N) x^N$$

IBP (extension of REDUZE_2, A.v. Manteuffel) gives

$$\sum_{N=0}^{\infty} D_{12}(N)x^N = e_1(x,\varepsilon)\hat{B}_1(x) + e_2(x,\varepsilon)\hat{B}_2(x) + \dots$$
$$+ e_i(x,\varepsilon)\hat{B}_i(x) + \dots e_{92}(x,\varepsilon)\hat{B}_{92}(x)$$

with $e_i(x,\varepsilon)$=rational expression in x and ε

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Note: the $\hat{B}_i(x)$ can be represented as power series

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Goal: Expand the 92 master integrals

$$B_i(N) = b_{-3}(N)\varepsilon^{-3} + b_{-2}(N)\varepsilon^{-2} + b_{-1}(N)\varepsilon^{-1} + b_0(N)\varepsilon^0 + \dots$$

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Take the coefficient of x^N

$$D_{12}(N) = [x^N]\Big(e_1(x,\varepsilon)\hat{B}_1(x) + e_2(x,\varepsilon)\hat{B}_2(x) + \dots + e_i(x,\varepsilon)\hat{B}_i(x) + \dots e_{92}(x,\varepsilon)\hat{B}_{92}(x)\Big)$$

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Toolbox 1: summation summation

Feynman integrals

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$\downarrow$ non-trivial transformations (DESY)

multiple sums

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Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

$$F(\varepsilon,N)=\int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||?$$

$$F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+F_0(N)\varepsilon^0+\dots$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times$$
$$\times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}$$

where

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

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$$||$$

$$\Big(\sum_{k=1}^{N} f_{-3}(N,k)\Big)\varepsilon^{-3} + \Big(\sum_{k=1}^{N} f_{-2}(N,k)\Big)\varepsilon^{-2} + \Big(\sum_{k=1}^{N} f_{-1}(N,k)\Big)\varepsilon^{-1} + \dots$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

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$$||$$

$$\Big(\sum_{k=1}^{N}f_{-3}(N,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{N}f_{-2}(N,k)\Big)\varepsilon^{-2}+\Big(\boxed{\sum_{k=1}^{N}f_{-1}(N,k)}\Big)\varepsilon^{-1}+\dots$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

where

$$S_a(N) = \sum_{i=1}^{N} \frac{\mathsf{sign}(a)^i}{i^a} \text{ and } \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

Simplify

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$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\ -(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\ +(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\ =\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\tfrac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

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↓ (summation package `Sigma.m`)

$$\begin{aligned}&\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}\\&\quad+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}\\&\quad+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$$\bigcap$$

$$\Big\{c_1\,\frac{1-4N}{N+1}+c_2\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}$$

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$$||$$ (recurrence finding and solving)

$$\Big(\frac{1}{12}-\frac{1}{8}\zeta_2\Big)\frac{1-4N}{N+1}+1\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \dots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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FIND all solutions expressible by indefinite nested products/sums
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3. Find a "closed form"

F(N)=combined solutions in terms of indefinite nested sums.

Sigma.m is based on difference ring/field theory

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$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \Gamma(-1-\varepsilon/2)\Gamma(-\varepsilon) \sum_{j=0}^{N} (-x)^{N-j} y^{N-j+\varepsilon/2} (1-y)^{-\varepsilon/2} \times z^{N-j+1} (1-z)^{-1-\varepsilon/2} (1-xz)^{j} (1-yz)^{\varepsilon}$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \Gamma(-1-\varepsilon/2)\Gamma(-\varepsilon) \sum_{j=0}^{N} (-x)^{N-j} y^{N-j+\varepsilon/2} (1-y)^{-\varepsilon/2}$$
$$\times\, z^{N-j+1}(1-z)^{-1-\varepsilon/2}(1-xz)^j(1-yz)^{\varepsilon}$$

$$||$$

$$\Gamma(-1-\varepsilon/2)\Gamma(-\varepsilon)\Gamma(1-\varepsilon/2) \sum_{j=0}^{N}\sum_{k=0}^{j} \frac{(-1)^{N-j+k}}{N-j+k+1}\binom{j}{k}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(n-\varepsilon)\Gamma(n+N-j+1+\varepsilon/2)\Gamma(n+N-j+k+2)}{n!\Gamma(n+N-j+2)\Gamma(n+N-j+k+2-\varepsilon/2)}$$

|| symbolic summation

|| (50 min)

$$
\begin{aligned}
&\frac{1}{\varepsilon^3}\Bigg[-\frac{4}{(N+1)^2}+\frac{8(-1)^N}{(N+1)^2}-4S_2-8S_{-2}\Bigg]+\frac{1}{\varepsilon^2}\Bigg[\frac{2(N-2)}{(N+1)^3}-\frac{4(-1)^N N}{(N+1)^3}-4S_1S_2\\
&+\frac{2(N-1)}{N+1}S_2-6S_3+\left(\frac{4(N-1)}{N+1}-8S_1\right)S_{-2}-4S_{-3}+4S_{2,1}+8S_{-2,1}\Bigg]\\
&+\frac{1}{\varepsilon}\Bigg[\frac{-N^2+N+9}{(N+1)^4}+\frac{2(-1)^N\left(N^2+N-3\right)}{(N+1)^4}+\Bigg(-\frac{3}{2(N+1)^2}+\frac{3(-1)^N}{(N+1)^2}-\frac{3}{2}S_2\\
&-3S_{-2}\Bigg)\zeta_2+\left(\frac{2(N-1)}{N+1}S_2-6S_3+4S_{2,1}+8S_{-2,1}\right)S_1-2S_1^2S_2+\frac{3(N-1)}{N+1}S_3\\
&+\left(-\frac{4(-1)^N}{(N+1)^2}+\frac{3-N^2}{(N+1)^2}\right)S_2+3S_2^2+2S_4+\Bigg(-\frac{8(-1)^N}{(N+1)^2}-\frac{2(N-1)}{N+1}-4S_1^2\\
&+\frac{4(N-1)S_1}{N+1}+4S_2\Bigg)S_{-2}+4S_{-2}^2+\left(\frac{2(N-1)}{N+1}-4S_1\right)S_{-3}+2S_{-4}+6S_{3,1}\\
&-\frac{2(N-1)}{N+1}S_{2,1}-\frac{4(N-1)}{N+1}S_{-2,1}+4S_{-2,2}+4S_{-3,1}-4S_{2,1,1}-8S_{-2,1,1}\Bigg]\\
&+\frac{(N-4)\left(N^2+4N+6\right)}{2(N+1)^5}+\Bigg[\frac{3(N-2)}{4(N+1)^3}-\frac{3(-1)^N N}{2(N+1)^3}+\frac{3(N-1)S_2}{4(N+1)}-\frac{3}{2}S_1S_2\\
&-\frac{9}{4}S_3+\left(\frac{3(N-1)}{2(N+1)}-3S_1\right)S_{-2}-\frac{3}{2}S_{-3}+\frac{3}{2}S_{2,1}+3S_{-2,1}\Bigg]\zeta_2+\Bigg(\frac{2(-1)^N-1}{2(N+1)^2}\\
&-\frac{1}{2}S_2-S_{-2}\Bigg)\zeta_3-\frac{(-1)^N}{(N+1)^5}\left(N^3+2N^2-2N-10\right)+\Bigg[\frac{4(-1)^N}{(N+1)^2}S_2+3S_2^2\\
&+\frac{N-1}{N+1}\left(3S_3-2S_{2,1}-4S_{-2,1}-S_2\right)+2S_4+6S_{3,1}+4S_{-2,2}+4S_{-3,1}-4S_{2,1,1}\\
&-8S_{-2,1,1}\Bigg]S_1+\left(\frac{N-1}{N+1}S_2-3S_3+2S_{2,1}+4S_{-2,1}\right)S_1^2+\Bigg(\frac{2(-1)^N(N+3)}{(N+1)^3}\\
&+\frac{N^3+N^2-3N-9}{2(N+1)^3}-\frac{4}{3}S_3-4S_{2,1}-12S_{-2,1}\Bigg)S_2-\frac{2}{3}S_1^3S_2-\frac{9}{2}S_5+4S_1S_{-2}^2\\
&+\left(\frac{6(-1)^N}{(N+1)^2}+\frac{5-3N^2}{2(N+1)^2}\right)S_3+\frac{N-1}{N+1}\Bigg[-2S_{-2}^2-\frac{3}{2}S_2^2-S_4+\left(2S_1-1\right)S_{-3}\\
&+\left(1-2S_1+2S_1^2-2S_2\right)S_{-2}-S_{-4}+S_{2,1}+2S_{-2,1}-3S_{3,1}-2S_{-2,2}-2S_{-3,1}\\
&+2S_{2,1,1}+4S_{-2,1,1}\Bigg]+\Bigg[\frac{4(-1)^N(N+3)}{(N+1)^3}+\left(\frac{8(-1)^N}{(N+1)^2}+4S_2\right)S_1-\frac{4}{3}S_1^3-\frac{8}{3}S_3\\
&+8S_{2,1}-4S_{-2,1}\Bigg]S_{-2}+\left(\frac{4(-1)^N}{(N+1)^2}-2S_1^2+2S_2\right)S_{-3}+2S_1S_{-4}-S_{-5}+4S_{2,3}\\
&-\frac{4(-1)^N}{(N+1)^2}\left(S_{2,1}+2S_{-2,1}\right)+8S_{2,-3}-3S_{4,1}-6S_{-2,3}+8S_{-2,-3}-2S_{-4,1}+2S_{2,2,1}\\
&-8S_{2,1,-2}-6S_{3,1,1}-12S_{-2,1,-2}-4S_{-2,2,1}-4S_{-3,1,1}+4S_{2,1,1,1}+8S_{-2,1,1,1}
\end{aligned}
$$

|| (50 min)

$$\frac{1}{\varepsilon^3}\left[-\frac{4}{(N+1)^2}+\frac{8(-1)^N}{(N+1)^2}-4S_2-8S_{-2}\right]+\frac{1}{\varepsilon^2}\left[\frac{2(N-2)}{(N+1)^3}-\frac{4(-1)^N N}{(N+1)^3}-4S_1S_2\right.$$
$$\left.+\frac{2(N-1)}{N+1}S_2-6S_3+\left(\frac{4(N-1)}{N+1}-8S_1\right)S_{-2}-4S_{-3}+4S_{2,1}+8S_{-2,1}\right]$$
$$+\frac{1}{\varepsilon}\left[\frac{-N^2+N+9}{(N+1)^4}+\frac{2(-1)^N\left(N^2+N-3\right)}{(N+1)^4}+\left(-\frac{3}{2(N+1)^2}+\frac{3(-1)^N}{(N+1)^2}-\frac{3}{2}S_2\right.\right.$$
$$\left.-3S_{-2}\right)\zeta_2+\left(\frac{2(N-1)}{N+1}S_2-6S_3+4S_{2,1}+8S_{-2,1}\right)S_1-2S_1^2S_2+\frac{3(N-1)}{N+1}S_3$$
$$+\left(-\frac{4(-1)^N}{(N+1)^2}+\frac{3-N^2}{(N+1)^2}\right)S_2+3S_2^2+2S_4+\left(-\frac{8(-1)^N}{(N+1)^2}-\frac{2(N-1)}{N+1}-4S_1^2\right.$$
$$\left.+\frac{4(N-1)S_1}{N+1}+4S_2\right)S_{-2}+4S_{-2}^2+\left(\frac{2(N-1)}{N+1}-4S_1\right)S_{-3}+2S_{-4}+6S_{3,1}$$
$$\left.-\frac{2(N-1)}{N+1}S_{2,1}-\frac{4(N-1)}{N+1}S_{-2,1}+4S_{-2,2}+4S_{-3,1}-4S_{2,1,1}-8S_{-2,1,1}\right]$$

$$\ldots\frac{3(N-1)S_2}{4(N+1)}-\frac{3}{2}S_1S_2$$
$$\ldots+3S_{-2,1}\Big]\zeta_2+\left(\frac{2(-1)^N-1}{2(N+1)^2}\right.$$
$$\ldots 10)+\left[\frac{4(-1)^N}{(N+1)^2}S_2+3S_{-2}^2\right.$$
$$\ldots S_{3,1}+4S_{-2,2}+4S_{-3,1}-4S_{2,1,1}$$
$$\ldots_{2,1}\Big)S_1^2+\left(\frac{2(-1)^N(N+3)}{(N+1)^3}\right.$$
$$\ldots-\frac{2}{3}S_1^3S_2-\frac{9}{2}S_5+4S_1S_{-2}^2$$
$$+\left(\frac{6(-1)^N}{(N+1)^2}+\frac{5-3N^2}{2(N+1)^2}\right)S_3+\frac{N-1}{N+1}\left[-2S_{-2}^2-\frac{3}{2}S_2^2-S_4+(2S_1-1)S_{-3}\right.$$
$$+\left(1-2S_1+2S_1^2-2S_2\right)S_{-2}-S_{-4}+S_{2,1}+2S_{-2,1}-3S_{3,1}-2S_{-2,2}-\ldots S_{-3,1}$$
$$\left.+2S_{2,1,1}+4S_{-2,1,1}\right]+\left[\frac{4(-1)^N(N+3)}{(N+1)^3}+\left(\frac{8(-1)^N}{(N+1)^2}+4S_2\right)S_1-\frac{4}{3}S_1^3-\frac{8}{3}S_3\right.$$
$$\left.+8S_{2,1}-4S_{-2,1}\right]S_{-2}+\left(\frac{4(-1)^N}{(N+1)^2}-2S_1^2+2S_2\right)S_{-3}+2S_1S_{-4}-S_{-5}+4S_{2,3}$$
$$-\frac{4(-1)^N}{(N+1)^2}\left(S_{2,1}+2S_{-2,1}\right)+8S_{2,-3}-3S_{4,1}-6S_{-2,3}+8S_{-2,-3}-2S_{-4,1}+2S_{2,2,1}$$
$$-8S_{2,1,-2}-6S_{3,1,1}-12S_{-2,1,-2}-4S_{-2,2,1}-4S_{-3,1,1}+4S_{2,1,1,1}+8S_{-2,1,1,1}$$

$$S_{-2,1,1,1}(N)=\sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{1}{l}}{k}}{j}}{i^2}$$

J.A.M. Vermaseren, 1998; J. Blümlein/S. Kurth, 1998

Goal: Expand the 92 master integrals

$$\left.\begin{array}{l} B_1(N) \\ B_2(N) \\ \vdots \\ B_{54}(N) \end{array}\right\} \text{54 by symbolic summation}$$

$$F(N) = \frac{\Gamma\left(-\frac{\varepsilon}{2}\right)^2 \Gamma(N+1)}{\Gamma\left(\frac{\varepsilon}{2}+N+2\right)} \int_0^1 dx \int_0^1 dy \int_0^1 dz \; x^{\varepsilon/2} y^{\varepsilon/2} \times$$

$$\times \quad (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} (1-yz)^{\varepsilon/2} (x+y-1)^N$$

$$F(N) = \frac{\Gamma\left(-\frac{\varepsilon}{2}\right)^2 \Gamma(N+1)}{\Gamma\left(\frac{\varepsilon}{2}+N+2\right)} \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{\varepsilon/2} y^{\varepsilon/2} \times$$

$$\times \quad (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} (1-yz)^{\varepsilon/2} (x+y-1)^N$$

$$||$$

$$\frac{2}{3\varepsilon+2}\left[\left(-1-N-\frac{\varepsilon}{2}\right) F_1 - 2F_2 + 2F_3\right]$$

with

$$F_1 = \sum_{j=0}^{N} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{-j+N} N! \Gamma(-\frac{3\varepsilon}{2}) \Gamma(1+N) \Gamma(-\frac{\varepsilon}{2}+m)}{(1+\frac{\varepsilon}{2}+j+m) j! m! n!} \times$$

$$\times \frac{\Gamma(-\frac{\varepsilon}{2}+n)\Gamma(1+\frac{\varepsilon}{2}+n)\Gamma(1-j+N)\Gamma(1+\frac{\varepsilon}{2}+m+n+N)}{(-j+N)!\Gamma(2+\frac{\varepsilon}{2}+N)\Gamma(2+\frac{\varepsilon}{2}-j+n+N)\Gamma(1-\varepsilon+m+n+N)},$$

$$F_2 = \sum_{k=0}^{\infty} \frac{e^{-\frac{3\varepsilon\gamma}{2}} \Gamma(-\varepsilon) \Gamma(-\frac{\varepsilon}{2}) \Gamma(1+N) \Gamma(-\frac{\varepsilon}{2}+k) \Gamma(1+\frac{\varepsilon}{2}+k+N)^2}{k! \Gamma(2+\frac{\varepsilon}{2}+N) \Gamma(1-\frac{\varepsilon}{2}+k+N) \Gamma(2+\frac{\varepsilon}{2}+k+N)},$$

$$F_3 = \sum_{j=0}^{N} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{-j+N} (1+\frac{\varepsilon}{2}+j) N! \Gamma(-\frac{3\varepsilon}{2})}{(1+\frac{\varepsilon}{2}+j+m) j! m! n!}$$

$$\frac{\Gamma(1+N)\Gamma(-\frac{\varepsilon}{2}+m)\Gamma(-\frac{\varepsilon}{2}+n)\Gamma(1+\frac{\varepsilon}{2}+n)\Gamma(1-j+N)\Gamma(1+\frac{\varepsilon}{2}+m+n+N)}{(-j+N)!\Gamma(2+\frac{\varepsilon}{2}+N)\Gamma(2+\frac{\varepsilon}{2}-j+n+N)\Gamma(1-\varepsilon+m+n+N)}$$

$$F(N) = \frac{\Gamma\big(-\frac{\varepsilon}{2}\big)^2\Gamma(N+1)}{\Gamma\big(\frac{\varepsilon}{2}+N+2\big)} \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^{\varepsilon/2} y^{\varepsilon/2} \times$$

$$\times \quad (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} (1-yz)^{\varepsilon/2} (x+y-1)^N$$

$$||$$

$$\frac{2}{3\varepsilon+2}\Big[\big(-1-N-\frac{\varepsilon}{2}\big) F_1 - 2F_2 + 2F_3\Big]$$

with

$$F_1 = \sum_{j=0}^{N}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{-j+N} N!\Gamma(-\frac{3\varepsilon}{2})\Gamma(1+N)\Gamma(-\frac{\varepsilon}{2}+m)}{(1+\frac{\varepsilon}{2}+j+m)j!m!n!} \times$$

$$\times \frac{\Gamma(-\frac{\varepsilon}{2}+n)\Gamma(1+\frac{\varepsilon}{2}+n)\Gamma(1-j+N)\Gamma(1+\frac{\varepsilon}{2}+m+n+N)}{(-j+N)!\Gamma(2+\frac{\varepsilon}{2}+N)\Gamma(2+\frac{\varepsilon}{2}-j+n+N)\Gamma(1-\varepsilon+m+n+N)},$$

$$F_2 = \sum_{k=0}^{\infty} \frac{e^{-\frac{3\varepsilon\gamma}{2}}\Gamma(-\varepsilon)\Gamma(-\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(1+\frac{\varepsilon}{2}+k+N)^2}{k!\Gamma(2+\frac{\varepsilon}{2}+N)\Gamma(1-\frac{\varepsilon}{2}+k+N)\Gamma(2+\frac{\varepsilon}{2}+k+N)},$$

$$F_3 = \sum_{j=0}^{N}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{-j+N}(1+\frac{\varepsilon}{2}+j)N!\Gamma(-\frac{3\varepsilon}{2})}{(1+\frac{\varepsilon}{2}+j+m)j!m!n!}$$

$$\frac{\Gamma(1+N)\Gamma(-\frac{\varepsilon}{2}+m)\Gamma(-\frac{\varepsilon}{2}+n)\Gamma(1+\frac{\varepsilon}{2}+n)\Gamma(1-j+N)\Gamma(1+\frac{\varepsilon}{2}+m+n+N)}{(-j+N)!\Gamma(2+\frac{\varepsilon}{2}+N)\Gamma(2+\frac{\varepsilon}{2}-j+n+N)\Gamma(1-\varepsilon+m+n+N)}$$

Summation yields the expansion up to ε^1. But: it is needed up to ε^4...

Toolbox 2: Symbolic integration

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \; x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Toolbox 2: Symbolic integration

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

$\downarrow$ Ablinger's package `MultiIntegrate.m`

$$a_0(\varepsilon,N)F(N) + a_1(\varepsilon,N)F(N+1) + a_2(\varepsilon,N)F(N+2) + a_3(\varepsilon,N)F(N+3) + a_4(\varepsilon,N)F(N+4) + a_5(\varepsilon,N)F(N+5) = 0$$

Toolbox 2: Symbolic integration

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

$\downarrow$ Ablinger's package `MultiIntegrate.m`

$$a_0(\varepsilon,N)F(N) + a_1(\varepsilon,N)F(N+1) + a_2(\varepsilon,N)F(N+2) + a_3(\varepsilon,N)F(N+3) + a_4(\varepsilon,N)F(N+4) + a_5(\varepsilon,N)F(N+5) = 0$$

- Based on a fine–tuned multi-variate Almkvist-Zeilberger implementation (with extra features)
  M. Apagodu and D. Zeilberger. *Adv. Appl. Math. (Special Regev issue)*, 37:139–152, 2006.
  J. Ablinger, Ph.D. Thesis, JKU, 2012, arXiv:1305.0687 [math-ph]
  J. Ablinger, J. Blümlein, M. Round and C. Schneider,PoS LL **2012** (2012) 050 [arXiv:1210.1685 [cs.SC]
- Total computation time: about 9 hours

Deriving the ε-expansion from the recurrence (with Sigma.m)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F(N)\Big] \\
&+a_1(\varepsilon, N)\Big[F(N+1)\Big] \\
&+ \\
&\vdots \\
&+a_d(\varepsilon, N)\Big[F(N+d)\Big]
\end{aligned}
$$

$$= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots$$

given (in terms of indefinite nested sums and products)

Deriving the ε-expansion from the recurrence (with `Sigma.m`)

$$a_0(\varepsilon,N)\Big[F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]$$

$$+a_1(\varepsilon,N)\Big[F(N+1)\Big]$$

$$+$$

$$\vdots$$

$$+a_d(\varepsilon,N)\Big[F(N+d)\Big]$$

$$=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots$$

given (in terms of indefinite nested sums and products)

Deriving the ε-expansion from the recurrence (with Sigma.m)

$$a_0(\varepsilon,N)\Big[F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]$$

$$+a_1(\varepsilon,N)\Big[F_{-3}(N+1)\varepsilon^{-3}+F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]$$

$$+$$

$$\vdots$$

$$+a_d(\varepsilon,N)\Big[F(N+d)\Big]$$

$$=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots$$

given (in terms of indefinite nested sums and products)

Deriving the ε-expansion from the recurrence (with `Sigma.m`)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_{-3}(N+1)\varepsilon^{-3}+F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_{-3}(N+d)\varepsilon^{-3}+F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Deriving the ε-expansion from the recurrence (with `Sigma.m`)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_{-3}(N+1)\varepsilon^{-3}+F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_{-3}(N+d)\varepsilon^{-3}+F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$
\boxed{a_0(0,N)F_{-3}(N)+a_1(0,N)F_{-3}(N+1)+\dots+a_d(0,N)F_{-3}(N+d)=h_{-3}(N)}
$$

Deriving the ε-expansion from the recurrence (with `Sigma.m`)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_{-3}(N+1)\varepsilon^{-3}+F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_{-3}(N+d)\varepsilon^{-3}+F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$
\boxed{a_0(0,N)F_{-3}(N)+a_1(0,N)F_{-3}(N+1)+\dots+a_d(0,N)F_{-3}(N+d)=h_{-3}(N)}
$$

REC solver: Using the initial values $F_{-3}(1),F_{-3}(2),\dots$ determines $F_{-3}(N)$ in terms of indefinite nested sums and products.

Deriving the ε-expansion from the recurrence (with `Sigma.m`)

$$\begin{aligned}
&a_0(\varepsilon,N)\Big[F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_{-3}(N+1)\varepsilon^{-3}+F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\vdots\\
&+a_d(\varepsilon,N)\Big[F_{-3}(N+d)\varepsilon^{-3}+F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)F_{-3}(N)+a_1(0,N)F_{-3}(N+1)+\dots+a_d(0,N)F_{-3}(N+d)=h_{-3}(N)}$$

Deriving the ε-expansion from the recurrence (with Sigma.m)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_{-3}(N+1)\varepsilon^{-3}+F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_{-3}(N+d)\varepsilon^{-3}+F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$
\boxed{a_0(0,N)F_{-3}(N)+a_1(0,N)F_{-3}(N+1)+\dots+a_d(0,N)F_{-3}(N+d)=h_{-3}(N)}
$$

$$
\begin{aligned}
&\ \ a_0(\varepsilon,N)\Big[F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \ \vdots\\
&+a_d(\varepsilon,N)\Big[F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad\qquad= h'_{-3}(N)\varepsilon^{-3}+h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon,N)\Big[F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad = \underbrace{h'_{-3}(N)}_{\mathbf{=0}}\varepsilon^{-3}+h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_{-2}(N+1)\varepsilon^{-2}+F_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\vdots\\
&+a_d(\varepsilon,N)\Big[F_{-2}(N+d)\varepsilon^{-2}+F_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

Now repeat for $F_{-2}(N), F_{-1}(N), \dots$

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$a_0(\varepsilon,N)F(N)+a_1(\varepsilon,N)F(N+1)+a_2(\varepsilon,N)F(N+2)$$
$$+a_3(\varepsilon,N)F(N+3)+a_4(\varepsilon,N)F(N+4)+a_5(\varepsilon,N)F(N+5)=0$$

$$+$$

$$\begin{aligned}
F(2) =& F_{-3}(2)\varepsilon^{-3}+F_{-2}(2)\varepsilon^{-2}+\dots+F_4(2)\varepsilon^4+\dots\\
F(3) =& F_{-3}(3)\varepsilon^{-3}+F_{-2}(3)\varepsilon^{-2}+\dots+F_4(3)\varepsilon^4+\dots\\
F(4) =& F_{-3}(4)\varepsilon^{-3}+F_{-2}(4)\varepsilon^{-2}+\dots+F_4(4)\varepsilon^4+\dots\\
F(5) =& F_{-3}(5)\varepsilon^{-3}+F_{-2}(5)\varepsilon^{-2}+\dots+F_4(5)\varepsilon^4+\dots\\
F(6) =& F_{-3}(6)\varepsilon^{-3}+F_{-2}(6)\varepsilon^{-2}+\dots+F_4(6)\varepsilon^4+\dots
\end{aligned}$$

$$\downarrow$$

$$F(N)=F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+\dots+F_4(N)\varepsilon^4+\dots$$

$$a_0(\varepsilon,N)F(N)+a_1(\varepsilon,N)F(N+1)+a_2(\varepsilon,N)F(N+2)$$
$$+a_3(\varepsilon,N)F(N+3)+a_4(\varepsilon,N)F(N+4)+a_5(\varepsilon,N)F(N+5)=0$$

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$$\begin{aligned}
F(2) =& F_{-3}(2)\varepsilon^{-3}+F_{-2}(2)\varepsilon^{-2}+\dots+F_4(2)\varepsilon^4+\dots\\
F(3) =& F_{-3}(3)\varepsilon^{-3}+F_{-2}(3)\varepsilon^{-2}+\dots+F_4(3)\varepsilon^4+\dots\\
F(4) =& F_{-3}(4)\varepsilon^{-3}+F_{-2}(4)\varepsilon^{-2}+\dots+F_4(4)\varepsilon^4+\dots\\
F(5) =& F_{-3}(5)\varepsilon^{-3}+F_{-2}(5)\varepsilon^{-2}+\dots+F_4(5)\varepsilon^4+\dots\\
F(6) =& F_{-3}(6)\varepsilon^{-3}+F_{-2}(6)\varepsilon^{-2}+\dots+F_4(6)\varepsilon^4+\dots
\end{aligned}$$

$$\downarrow$$

$$F(N)=F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+\dots+F_4(N)\varepsilon^4+\dots$$

The coefficients:

$$
\begin{aligned}
a_0(\varepsilon, N) = {} & (N+1)(N+2)\big(8\varepsilon^{10} + 104\varepsilon^9(N+3) + 4\varepsilon^8\big(96N^2 + 601N + 887\big) \\
& + 4\varepsilon^7\big(12N^3 + 414N^2 + 1583N + 1393\big) \\
& - 8\varepsilon^6\big(264N^4 + 2436N^3 + 8643N^2 + 14518N + 9947\big) \\
& - 16\varepsilon^5\big(156N^5 + 1690N^4 + 6847N^3 + 12661N^2 + 9537N + 717\big) \\
& + 32\varepsilon^4\big(68N^6 + 1158N^5 + 8155N^4 + 30114N^3 + 61712N^2 + 67616N + 31693\big) \\
& + 64\varepsilon^3\big(40N^7 + 560N^6 + 2755N^5 + 3729N^4 - 14194N^3 - 61920N^2 - 89140N - 46600\big) \\
& - 128\varepsilon^2(N+2)\big(12N^7 + 254N^6 + 2249N^5 + 10758N^4 + 30173N^3 + 50610N^2 \\
& + 49122N + 22706\big) \\
& + 256\varepsilon(N+2)^2(N+3)(N+4)\big(44N^4 + 501N^3 + 2044N^2 + 3455N + 1976\big) \\
& - 512(N+1)(N+2)^3(N+3)^2(N+4)\big(6N^2 + 47N + 95\big)\big), \\
a_1(\varepsilon, N) = {} & (N+2)\big(-22\varepsilon^{11} - 2\varepsilon^{10}(157N + 435) - \varepsilon^9\big(1500N^2 + 8611N + 11745\big) \\
& - \varepsilon^8\big(2548N^3 + 22936N^2 + 63597N + 54229\big) \\
& + 4\varepsilon^7\big(266N^4 + 1857N^3 + 6065N^2 + 14351N + 15987\big) \\
& + 8\varepsilon^6\big(994N^5 + 12961N^4 + 67246N^3 + 174692N^2 + 226821N + 116092\big) \\
& + 16\varepsilon^5\big(336N^6 + 5348N^5 + 33569N^4 + 104918N^3 + 165290N^2 + 108259N + 6100\big)
\end{aligned}
$$

The coefficients:

$$
\begin{aligned}
a_2(\varepsilon, N) = \big(&12\varepsilon^{12} + 12\varepsilon^{11}(17N + 45) + 2\varepsilon^{10}\big(620N^2 + 3553N + 4795\big)\\
&+ 2\varepsilon^9\big(1504N^3 + 14190N^2 + 41901N + 38907\big)\\
&+ 4\varepsilon^8\big(172N^4 + 4983N^3 + 30942N^2 + 69119N + 50850\big)\\
&- 4\varepsilon^7\big(1996N^5 + 24056N^4 + 113313N^3 + 269119N^2 + 337198N + 185290\big)\\
&- 16\varepsilon^6\big(450N^6 + 8210N^5 + 59749N^4 + 227386N^3 + 486841N^2 + 563176N + 275664\big)\\
&+ 16\varepsilon^5\big(340N^7 + 4314N^6 + 19137N^5 + 25532N^4 - 55105N^3 - 206516N^2 - 191528N\\
&- 23458\big)\\
&+ 32\varepsilon^4\big(140N^8 + 2940N^7 + 26550N^6 + 139926N^5 + 493839N^4 + 1240186N^3\\
&+ 2161699N^2 + 2304248N + 1100084\big)\\
&+ 64\varepsilon^3\big(4N^9 + 506N^8 + 8651N^7 + 63510N^6 + 236215N^5 + 395334N^4 - 105413N^3\\
&- 1551017N^2 - 2362944N - 1217770\big)\\
&- 128\varepsilon^2(N + 3)\big(12N^9 + 314N^8 + 3782N^7 + 29105N^6 + 160727N^5 + 640273N^4\\
&+ 1750874N^3 + 3052505N^2 + 3017094N + 1276604\big)\\
&+ 256\varepsilon(N + 2)(N + 3)^2(N + 4)\big(26N^6 + 825N^5 + 8967N^4 + 46529N^3 + 125411N^2\\
&+ 168628N + 88652\big)\\
&512(N+1)(N+2)^2(N+3)^3(N+4)^2(6N^3+98N^2+459N+655)\big)
\end{aligned}
$$

The coefficients:

$$
\begin{aligned}
a_3(\varepsilon, N) = \big(&- 64\varepsilon^{12} - 8\varepsilon^{11}(113N + 298) - 8\varepsilon^{10}\big(519N^2 + 2948N + 3896\big) \\
&- 4\varepsilon^9\big(1444N^3 + 13839N^2 + 39746N + 34305\big) \\
&+ 4\varepsilon^8\big(1948N^4 + 17868N^3 + 63837N^2 + 112966N + 84655\big) \\
&+ 16\varepsilon^7\big(1456N^5 + 20460N^4 + 112365N^3 + 304963N^2 + 412258N + 221769\big) \\
&- 8\varepsilon^6\big(320N^6 + 2050N^5 + 4192N^4 + 27408N^3 + 174901N^2 + 411759N + 324872\big) \\
&- 16\varepsilon^5\big(1756N^7 + 33154N^6 + 265889N^5 + 1186719N^4 + 3218059N^3 + 5349388N^2 \\
&+ 5071913N + 2113696\big) \\
&+ 32\varepsilon^4\big(188N^8 + 4802N^7 + 59527N^6 + 439922N^5 + 2025336N^4 + 5813984N^3 \\
&+ 10076450N^2 + 9621283N + 3878602\big) \\
&+ 64\varepsilon^3\big(140N^9 + 2768N^8 + 22500N^7 + 99545N^6 + 287700N^5 + 723136N^4 \\
&+ 1854572N^3 + 3714620N^2 + 4272517N + 2031600\big) \\
&- 128\varepsilon^2\big(24N^{10} + 830N^9 + 14362N^8 + 152630N^7 + 1053620N^6 + 4834279N^5 \\
&+ 14824351N^4 + 29964399N^3 + 38244797N^2 + 27875896N + 8824032\big) \\
&+ 256\varepsilon(N + 2)(N + 3)(N + 4)\big(118N^7 + 2639N^6 + 24247N^5 + 118311N^4 + 329565N^3 \\
&+ 520306N^2 + 426076N + 136854\big) \\
&- 512(N + 1)(N + 2)^2(N + 3)^2(N + 4)^2(N + 5)\big(12N^3 + 97N^2 + 230N + 144\big)\big)
\end{aligned}
$$

The coefficients:

$$
\begin{aligned}
a_4(\varepsilon, N) = \big(&64\varepsilon^{12} + 192\varepsilon^{11}(5N+14) + 16\varepsilon^{10}\big(297N^2 + 1769N + 2451\big) \\
&+ 16\varepsilon^{9}\big(453N^3 + 4462N^2 + 13094N + 11244\big) \\
&- 8\varepsilon^{8}\big(1084N^4 + 11117N^3 + 47258N^2 + 103981N + 94650\big) \\
&- 8\varepsilon^{7}\big(3304N^5 + 51138N^4 + 311957N^3 + 948722N^2 + 1440105N + 858544\big) \\
&+ 16\varepsilon^{6}\big(420N^6 + 5507N^5 + 36275N^4 + 169650N^3 + 536911N^2 + 952507N + 694370\big) \\
&+ 16\varepsilon^{5}\big(1828N^7 + 38868N^6 + 353301N^5 + 1801014N^4 + 5604391N^3 + 10664390N^2 \\
&+ 11433064N + 5260048\big) \\
&- 32\varepsilon^{4}\big(316N^8 + 8356N^7 + 105800N^6 + 802421N^5 + 3836854N^4 + 11588223N^3 \\
&+ 21401558N^2 + 22066744N + 9745752\big) \\
&- 64\varepsilon^{3}\big(116N^9 + 2424N^8 + 19923N^7 + 82966N^6 + 208191N^5 + 530980N^4 + 1847484N^3 \\
&+ 4687014N^2 + 6120858N + 3111104\big) \\
&+ 128\varepsilon^{2}\big(24N^{10} + 826N^9 + 14897N^8 + 172000N^7 + 1314686N^6 + 6710299N^5 \\
&+ 22873183N^4 + 51298261N^3 + 72551278N^2 + 58573022N + 20544948\big) \\
&- 256\varepsilon(N+2)(N+3)\big(106N^8 + 3278N^7 + 42903N^6 + 310942N^5 + 1366350N^4 \\
&+ 3729418N^3 + 6173159N^2 + 5657732N + 2191212\big) \\
&+ 512(N+1)(N+2)^2(N+3)^2(N+4)(N+5)(N+6)(12N^3 + 131N^2 + 396N + 431)\big)
\end{aligned}
$$

The coefficients:

$$
\begin{aligned}
a_5(\varepsilon,N) = (N+5)\big(&-128\varepsilon^{11} - 128\varepsilon^{10}(11N+26) - 32\varepsilon^{9}\big(115N^2+592N+647\big)\\
&+32\varepsilon^{8}\big(63N^3+430N^2+1665N+2384\big)\\
&+16\varepsilon^{7}\big(714N^4+7881N^3+33802N^2+66225N+47654\big)\\
&-16\varepsilon^{6}\big(234N^5+2444N^4+13989N^3+50862N^2+104083N+87848\big)\\
&-16\varepsilon^{5}\big(580N^6+10181N^5+76586N^4+319207N^3+772120N^2+1012046N+547832\big)\\
&+16\varepsilon^{4}\big(244N^7+5456N^6+61605N^5+401216N^4+1536277N^3+3408574N^2\\
&+4066436N+2026928\big)\\
&+64\varepsilon^{3}\big(26N^8+357N^7+583N^6-11139N^5-65193N^4-120264N^3+11864N^2\\
&+272830N+222624\big)\\
&-64\varepsilon^{2}(N+3)\big(12N^8+298N^7+4684N^6+49024N^5+306907N^4+1122441N^3\\
&+2350650N^2+2607576N+1185072\big)\\
&+256\varepsilon(N+2)(N+3)\big(25N^7+743N^6+8856N^5+55358N^4+197497N^3+404131N^2\\
&+439902N+196128\big)\\
&-256(N+1)(N+2)^2(N+3)^2(N+4)(N+6)(N+7)\big(6N^2+35N+54\big)\big)
\end{aligned}
$$

$$a_0(\varepsilon, N)F(N) + a_1(\varepsilon, N)F(N+1) + a_2(\varepsilon, N)F(N+2)$$
$$+ a_3(\varepsilon, N)F(N+3) + a_4(\varepsilon, N)F(N+4) + a_5(\varepsilon, N)F(N+5) = 0$$

$$+$$

$$F(2) = F_{-3}(2)\varepsilon^{-3} + F_{-2}(2)\varepsilon^{-2} + \dots + F_4(2)\varepsilon^4 + \dots$$
$$F(3) = F_{-3}(3)\varepsilon^{-3} + F_{-2}(3)\varepsilon^{-2} + \dots + F_4(3)\varepsilon^4 + \dots$$
$$F(4) = F_{-3}(4)\varepsilon^{-3} + F_{-2}(4)\varepsilon^{-2} + \dots + F_4(4)\varepsilon^4 + \dots$$
$$F(5) = F_{-3}(5)\varepsilon^{-3} + F_{-2}(5)\varepsilon^{-2} + \dots + F_4(5)\varepsilon^4 + \dots$$
$$F(6) = F_{-3}(6)\varepsilon^{-3} + F_{-2}(6)\varepsilon^{-2} + \dots + F_4(6)\varepsilon^4 + \dots$$

$$\downarrow$$

$$F(N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + \dots + F_4(N)\varepsilon^4 + \dots$$

The initial values

$$
\begin{aligned}
F(2,\varepsilon) = {} & \frac{20}{27\varepsilon^3} - \frac{40}{27\varepsilon^2} + \frac{\frac{1393}{486} + \frac{5\zeta_2}{18}}{\varepsilon} - \frac{9601}{1944} - \frac{5\zeta_2}{9} + \frac{49\zeta_3}{54} \\
& + \varepsilon\Big(\frac{565297}{69984} + \frac{1393\zeta_2}{1296} + \frac{1151\zeta_2^2}{1440} - \frac{137\zeta_3}{54}\Big) \\
& + \varepsilon^2\Big(-\frac{1150003}{93312} - \frac{9601\zeta_2}{5184} - \frac{1619\zeta_2^2}{720} + \frac{17831\zeta_3}{3888} + \frac{49\zeta_2\zeta_3}{144} + \frac{265\zeta_5}{72}\Big) \\
& + \varepsilon^3\Big(\frac{184376401}{10077696} + \frac{565297\zeta_2}{186624} + \frac{420979\zeta_2^2}{103680} + \frac{449843\zeta_2^3}{241920} - \frac{103607\zeta_3}{15552} \\
& - \frac{137}{144}\zeta_2\zeta_3 - \frac{259\zeta_3^2}{864} - \frac{191\zeta_5}{18}\Big) + O(\varepsilon^4), \\
F(3,\varepsilon) = {} & \frac{1}{6\varepsilon^3} - \frac{11}{48\varepsilon^2} + \frac{\frac{703}{3456} + \frac{\zeta_2}{16}}{\varepsilon} - \frac{9773}{9216} - \frac{11\zeta_2}{128} + \frac{41\zeta_3}{48} \\
& + \varepsilon\Big(\frac{2157295}{1990656} + \frac{703\zeta_2}{9216} + \frac{979\zeta_2^2}{1280} - \frac{931\zeta_3}{384}\Big) \\
& + \varepsilon^2\Big(-\frac{60535183}{15925248} - \frac{9773\zeta_2}{24576} - \frac{22289\zeta_2^2}{10240} + \frac{137591\zeta_3}{27648} + \frac{41\zeta_2\zeta_3}{128} + \frac{1201\zeta_5}{320}\Big) \\
& + \varepsilon^3\Big(\frac{2116767175}{1146617856} + \frac{2157295\zeta_2}{5308416} + \frac{3298669\zeta_2^2}{737280} + \frac{406711\zeta_2^3}{215040} - \frac{550229\zeta_3}{73728} \\
& - \frac{931\zeta_2\zeta_3}{1024} - \frac{239\zeta_3^2}{768} - \frac{27611\zeta_5}{2560}\Big) + O(\varepsilon^4),
\end{aligned}
$$

The initial values

$$
\begin{aligned}
F(4,\varepsilon) = {} & \frac{64}{225\varepsilon^3} - \frac{1748}{3375\varepsilon^2} + \frac{\frac{102181}{101250} + \frac{8\zeta_2}{75}}{\varepsilon} - \frac{5738207}{2430000} - \frac{437\zeta_2}{2250} + \frac{196\zeta_3}{225} \\
& + \varepsilon\Big(\frac{1681164919}{486000000} + \frac{102181\zeta_2}{270000} + \frac{583\zeta_2^2}{750} - \frac{17059\zeta_3}{6750}\Big) \\
& + \varepsilon^2\Big(-\frac{423112175849}{58320000000} - \frac{5738207\zeta_2}{6480000} - \frac{407231\zeta_2^2}{180000} + \frac{4288637\zeta_3}{810000} + \frac{49\zeta_2\zeta_3}{150} + \frac{1412\zeta_5}{375}\Big) \\
& + \varepsilon^3\Big(\frac{157023072517301}{20995200000000} + \frac{1681164919\zeta_2}{1296000000} + \frac{102416383\zeta_2^2}{21600000} + \frac{239203\zeta_2^3}{126000} \\
& - \frac{155753563\zeta_3}{19440000} - \frac{17059\zeta_2\zeta_3}{18000} - \frac{14\zeta_3^2}{45} - \frac{499097\zeta_5}{45000}\Big) + O(\varepsilon^4), \\
F(5,\varepsilon) = {} & \frac{2}{27\varepsilon^3} - \frac{17}{162\varepsilon^2} + \frac{\frac{7583}{97200} + \frac{\zeta_2}{36}}{\varepsilon} - \frac{1666837}{1296000} - \frac{17\zeta_2}{432} + \frac{113\zeta_3}{108} \\
& + \varepsilon\Big(\frac{187423951}{155520000} + \frac{7583\zeta_2}{259200} + \frac{2707\zeta_2^2}{2880} - \frac{19769\zeta_3}{6480}\Big) \\
& + \varepsilon^2\Big(-\frac{6827006887}{1244160000} - \frac{1666837\zeta_2}{3456000} - \frac{474031\zeta_2^2}{172800} + \frac{5247499\zeta_3}{777600} + \frac{113\zeta_2\zeta_3}{288} + \frac{3361\zeta_5}{720}\Big) \\
& + \varepsilon^3\Big(\frac{40517946316703}{20155392000000} + \frac{187423951\zeta_2}{414720000} + \frac{125902061\zeta_2^2}{20736000} + \frac{227531\zeta_2^3}{96768} - \frac{321933739\zeta_3}{31104000} \\
& - \frac{19769\zeta_2\zeta_3}{17280} - \frac{671\zeta_3^2}{1728} - \frac{118121\zeta_5}{8640}\Big) + O(\varepsilon^4),
\end{aligned}
$$

The initial values

$$
\begin{aligned}
F(6,\varepsilon) = &+\frac{22}{147\varepsilon^3} - \frac{535}{2058\varepsilon^2} + \frac{\frac{630043}{1234800} + \frac{11\zeta_2}{196}}{\varepsilon} - \frac{1949958721}{871274880} - \frac{535\zeta_2}{5488} + \frac{775\zeta_3}{588} \\
&+ \varepsilon\Big(\frac{1873033251331}{677658240000} + \frac{630043\zeta_2}{3292800} + \frac{3709\zeta_2^2}{3136} - \frac{321149\zeta_3}{82320}\Big) \\
&+ \varepsilon^2\Big(-\frac{181315912291471}{20910597120000} - \frac{1949958721\zeta_2}{2323399680} - \frac{7694201\zeta_2^2}{2195200} + \frac{86628433\zeta_3}{9878400} + \frac{775\zeta_2\zeta_3}{1568} \\
&+ \frac{22931\zeta_5}{3920}\Big) \\
&+ \varepsilon^3\Big(\frac{7300479289785352015 3}{129102026618880000 00} + \frac{1873033251331\zeta_2}{1807088640000} + \frac{2075932177\zeta_2^2}{263424000} + \frac{7763069\zeta_2^3}{2634240} \\
&- \frac{2376849605591\zeta_3}{174254976000} - \frac{321149\zeta_2\zeta_3}{219520} - \frac{4573\zeta_3^2}{9408} - \frac{1911379\zeta_5}{109760}\Big) + O(\varepsilon^4)
\end{aligned}
$$

$$a_0(\varepsilon,N)F(N)+a_1(\varepsilon,N)F(N+1)+a_2(\varepsilon,N)F(N+2)$$
$$+a_3(\varepsilon,N)F(N+3)+a_4(\varepsilon,N)F(N+4)+a_5(\varepsilon,N)F(N+5)=0$$

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$$\begin{aligned}
F(2) =& F_{-3}(2)\varepsilon^{-3}+F_{-2}(2)\varepsilon^{-2}+\dots+F_4(2)\varepsilon^4+\dots\\
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F(5) =& F_{-3}(5)\varepsilon^{-3}+F_{-2}(5)\varepsilon^{-2}+\dots+F_4(5)\varepsilon^4+\dots\\
F(6) =& F_{-3}(6)\varepsilon^{-3}+F_{-2}(6)\varepsilon^{-2}+\dots+F_4(6)\varepsilon^4+\dots
\end{aligned}$$

$$\downarrow$$

$$F(N)=F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+\dots+F_4(N)\varepsilon^4+\dots$$

We get:

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

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$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

$$F_{-2}(N) = -\frac{4(-1)^N\left(3N^3+18N^2+31N+18\right)}{3(N+1)^3(N+2)^2} - \frac{4\left(6N^3+32N^2+51N+26\right)}{3(N+1)^3(N+2)^2}$$

We get:

$$
\begin{aligned}
F_{-3}(N) =& \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)} \\
F_{-2}(N) =& -\frac{4(-1)^N\big(3N^3+18N^2+31N+18\big)}{3(N+1)^3(N+2)^2} - \frac{4\big(6N^3+32N^2+51N+26\big)}{3(N+1)^3(N+2)^2} \\
F_{-1}(N) =& (-1)^N\Big(\frac{2\big(9N^5+81N^4+295N^3+533N^2+500N+204\big)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)}\Big) \\
&+ \frac{2\big(18N^5+150N^4+490N^3+755N^2+536N+132\big)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)} \\
&+ \Big(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)}\Big)S_2(N) \\
&+ \Big(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)}\Big)S_{-2}(N)
\end{aligned}
$$

We get:

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$$\begin{aligned}F_{-1}(N) =&(-1)^N\Big(\frac{2\big(9N^5+81N^4+295N^3+533N^2+500N+204\big)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)}\Big)\\ &+ \frac{2\big(18N^5+150N^4+490N^3+755N^2+536N+132\big)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)}\\ &+ \Big(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)}\Big)S_2(N)\\ &+ \Big(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)}\Big)S_{-2}(N)\end{aligned}$$

$$F_0(N) = \dots$$

$$F_1(N) = \dots$$

$$F_2(N) = \dots$$

$$F_3(N) = \dots$$

$$F_4(N) = \dots$$

We get:

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

$$F_{-2}(N) = -\frac{4(-1)^N\big(3N^3+18N^2+31N+18\big)}{3(N+1)^3(N+2)^2} - \frac{4\big(6N^3+32N^2+51N+26\big)}{3(N+1)^3(N+2)^2}$$

$$\begin{aligned}F_{-1}(N) =& (-1)^N\Big(\frac{2\big(9N^5+81N^4+295N^3+533N^2+500N+204\big)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)}\Big)\\ &+ \frac{2\big(18N^5+150N^4+490N^3+755N^2+536N+132\big)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)}\\ &+ \Big(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)}\Big)S_2(N)\\ &+ \Big(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)}\Big)S_{-2}(N)\end{aligned}$$

$F_0(N) = \dots$

$F_1(N) = \dots$

$F_2(N) = \dots$

$F_3(N) = \dots$

$F_4(N) = \dots$

819 S-sums (algebraically independent!) pop up

We get (in about 2 hours):

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

$$F_{-2}(N) = -\frac{4(-1)^N\big(3N^3+18N^2+31N+18\big)}{3(N+1)^3(N+2)^2} - \frac{4\big(6N^3+32N^2+51N+26\big)}{3(N+1)^3(N+2)^2}$$

$$\begin{aligned}F_{-1}(N) =& (-1)^N\Big(\frac{2\big(9N^5+81N^4+295N^3+533N^2+500N+204\big)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)}\Big)\\ &+ \frac{2\big(18N^5+150N^4+490N^3+755N^2+536N+132\big)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)}\\ &+ \Big(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)}\Big)S_2(N)\\ &+ \Big(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)}\Big)S_{-2}(N)\end{aligned}$$

$$F_0(N) = \dots$$
$$F_1(N) = \dots$$
$$F_2(N) = \dots$$
$$F_3(N) = \dots$$
$$F_4(N) = \dots$$

819 S-sums (algebraically independent!) pop up, e.g.,

$$S_{1,2,1,1,1,1}\Big(\tfrac{1}{2},-1,1,1,1,1,N\Big) = \sum_{i=1}^{N}\frac{1}{2^i i}\sum_{j=1}^{i}\frac{(-1)^j}{j^2}\sum_{k=1}^{j}\frac{1}{k}\sum_{l=1}^{k}\frac{1}{l}\sum_{m=1}^{l}\frac{1}{m}\sum_{n=1}^{m}\frac{1}{n}$$

S. Moch, P. Uwer, S. Weinzierl, 2002

Goal: Expand the 92 master integrals

$$\left.\begin{matrix} B_1(N) \\ B_2(N) \\ \vdots \\ B_{54}(N) \end{matrix}\right\}$$ 54 by symbolic summation

$$\left.\begin{matrix} B_{55}(N) \\ B_{56}(N) \\ \vdots \\ B_{62}(N) \end{matrix}\right\}$$ 8 by symbolic integration and recurrence solving

Goal: Expand the 92 master integrals

$$\left.\begin{array}{l} B_1(N) \\ B_2(N) \\ \vdots \\ B_{54}(N) \end{array}\right\}$$ 54 by symbolic summation

$$\left.\begin{array}{l} B_{55}(N) \\ B_{56}(N) \\ \vdots \\ B_{62}(N) \end{array}\right\}$$ 8 by symbolic integration and recurrence solving

$$\left.\begin{array}{l} B_{63}(N) \\ B_{64}(N) \\ \vdots \\ B_{92}(N) \end{array}\right\}$$ no suitable representations for symbolic summation/integration

Goal: Expand the 92 master integrals

$$\left.\begin{array}{l} B_1(N) \\ B_2(N) \\ \vdots \\ B_{54}(N) \end{array}\right\} \text{54 by symbolic summation}$$

$$\left.\begin{array}{l} B_{55}(N) \\ B_{56}(N) \\ \vdots \\ B_{62}(N) \end{array}\right\} \text{8 by symbolic integration and recurrence solving}$$

$$\left.\begin{array}{l} \hat{B}_{63}(x) \\ \hat{B}_{64}(x) \\ \vdots \\ \hat{B}_{92}(x) \end{array}\right\} B_i(N) \to \hat{B}_i(x) = \sum_{N=0}^{\infty} B_i(N) x^N$$

Goal: Expand the 92 master integrals

$$\left.\begin{array}{l} B_1(N) \\ B_2(N) \\ \vdots \\ B_{54}(N) \end{array}\right\}$$ 54 by symbolic summation

$$\left.\begin{array}{l} B_{55}(N) \\ B_{56}(N) \\ \vdots \\ B_{62}(N) \end{array}\right\}$$ 8 by symbolic integration and recurrence solving

$$\left.\begin{array}{l} \hat{B}_{63}(x) \\ \hat{B}_{64}(x) \\ \vdots \\ \hat{B}_{92}(x) \end{array}\right\}$$ `REDUZE_2` (A.v. Manteuffel) produces a (recursively organized) coupled differential system

Goal: Expand the 92 master integrals

$$\left.\begin{array}{l} B_1(N) \\ B_2(N) \\ \vdots \\ B_{54}(N) \end{array}\right\} \text{54 by symbolic summation}$$

$$\left.\begin{array}{l} B_{55}(N) \\ B_{56}(N) \\ \vdots \\ B_{62}(N) \end{array}\right\} \text{8 by symbolic integration and recurrence solving}$$

$$\left.\begin{array}{l} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \vdots \\ \hat{I}_{32}(x) \end{array}\right\} \begin{array}{l} \texttt{REDUZE\_2} \text{ (A.v. Manteuffel) produces a} \\ \text{(recursively organized) coupled differential system} \end{array}$$

Toolbox 3: Solving coupled differential equations

The coupled system for $\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)$

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\hat{R}_2(x) = \frac{(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x}\hat{B}_1(x) - \frac{(\varepsilon+2)(3\varepsilon+4)\left(19\varepsilon^2+36\varepsilon+16\right)}{16\varepsilon(5\varepsilon+6)(x-1)x}\hat{B}_2(x) - \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x}\hat{B}_3(x) - \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x}\hat{B}_4(x)$$

$\boxed{\hat{B}_1(x)}, \hat{B}_2(x), \hat{B}_3(x)$ have been solved with symbolic summation.

$$
\begin{aligned}
D_x \hat{I}_1(x) = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) - \tfrac{2}{(x-1)x} \hat{I}_2(x) \\
&+ \tfrac{1}{(x-1)x} \hat{B}_1(x) + \dots
\end{aligned}
$$

$$
\begin{aligned}
D_x \hat{I}_2(x) = &\frac{\left(3(\varepsilon+4)^2-22(\varepsilon+4)+40\right)}{4(x-1)} \hat{I}_1(x) \\
&+ \tfrac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x} \hat{I}_2(x) - \tfrac{(\varepsilon+1)}{2(x-1)} \hat{I}_3(x) \\
&+ \frac{1}{4} \frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots
\end{aligned}
$$

$$
\begin{aligned}
D_x \hat{I}_3(x) = &- \frac{\left(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\right)}{4(x-1)x} \hat{I}_1(x) \\
&+ \tfrac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x} \hat{I}_2(x) - \tfrac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x} \hat{I}_3(x) \\
&- \frac{1}{4} \frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots
\end{aligned}
$$

$$D_x\hat{I}_1(x) = -\frac{(-\varepsilon+x-1)}{(x-1)x}\hat{I}_1(x) - \frac{2}{(x-1)x}\hat{I}_2(x) + \frac{1}{(x-1)x}\hat{B}_1(x) + \dots$$

$$\begin{aligned} D_x\hat{I}_2(x) = &\frac{\left(3(\varepsilon+4)^2-22(\varepsilon+4)+40\right)}{4(x-1)}\hat{I}_1(x) \\ &+ \frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x) \\ &+ \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

$$\begin{aligned} D_x\hat{I}_3(x) = &-\frac{\left(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\right)}{4(x-1)x}\hat{I}_1(x) \\ &+ \frac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x}\hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x}\hat{I}_3(x) \\ &- \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

Step 1: From a DE system to a REC system

$$\begin{aligned} D_x \hat{I}_1(x) = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) \\ &- \tfrac{2}{(x-1)x} \hat{I}_2(x) \\ &+ \tfrac{1}{(x-1)x} \hat{B}_1(x) + \dots \end{aligned}$$

Step 1: From a DE system to a REC system

$$\begin{aligned}(x-1)xD_x\hat{I}_1(x) = &-(-\varepsilon+x-1)\hat{I}_1(x)\\ &-2\hat{I}_2(x)\\ &+\hat{B}_1(x)+\dots\end{aligned}$$

Step 1: From a DE system to a REC system

$$
\begin{aligned}
(x-1)xD_x\sum_{N=0}^{\infty}I_1(N)x^N = &-(-\varepsilon+x-1)\sum_{N=0}^{\infty}I_1(N)x^N\\
&-2\sum_{N=0}^{\infty}I_2(N)x^N\\
&+\sum_{N=0}^{\infty}B_1(N)x^N+\dots
\end{aligned}
$$

Step 1: From a DE system to a REC system

$$
\begin{aligned}
(x-1)x\sum_{N=1}^{\infty} I_1(N)Nx^{N-1} = &-(-\varepsilon+x-1)\sum_{N=0}^{\infty} I_1(N)x^N \\
&-2\sum_{N=0}^{\infty} I_2(N)x^N \\
&+\sum_{N=0}^{\infty} B_1(N)x^N + \ldots
\end{aligned}
$$

Step 1: From a DE system to a REC system

$$(x-1)x\sum_{N=1}^{\infty} I_1(N)Nx^{N-1} = -\left(-\varepsilon+x-1\right)\sum_{N=0}^{\infty} I_1(N)x^N$$
$$-2\sum_{N=0}^{\infty} I_2(N)x^N$$
$$+\sum_{N=0}^{\infty} B_1(N)x^N + \dots$$

$\downarrow$ Nth coefficient

$$NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)=B_1(N)+\dots$$

$$NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)$$
$$=B_1(N)+\dots$$

$$\begin{aligned}D_x\hat{I}_2(x) &= \frac{\left(3(\varepsilon+4)^2-22(\varepsilon+4)+40\right)}{4(x-1)}\hat{I}_1(x)\\ &+\frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x)-\frac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x)\\ &+\frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x)+\dots\end{aligned}$$

$$\begin{aligned}D_x\hat{I}_3(x) &= -\frac{\left(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\right)}{4(x-1)x}\hat{I}_1(x)\\ &+\frac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x}\hat{I}_2(x)-\frac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x}\hat{I}_3(x)\\ &-\frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x)+\dots\end{aligned}$$

A coupled system of difference equations

$$
\begin{aligned}
NI_1(N-1) - (\varepsilon + N + 1)I_1(N) + 2I_2(N)& \\
=&B_1(N) + \dots
\end{aligned}
$$

$$
\begin{aligned}
2(\varepsilon + 2N + 2)I_2(N) - 2(3\varepsilon + 2N + 1)I_2(N-1)& \\
+ \varepsilon(3\varepsilon + 2)I_1(N-1) - 2(\varepsilon + 1)I_3(N-1)& \\
=&(5\varepsilon + 4)B_1(N) - \frac{2(\varepsilon + 1)(3\varepsilon + 4)}{5\varepsilon + 6}B_1(N-1) + \dots
\end{aligned}
$$

$$
\begin{aligned}
4(\varepsilon - N)I_3(N) - 2\varepsilon(3\varepsilon + 2)I_1(N) + \varepsilon(3\varepsilon + 2)I_1(N-1)& \\
- 2(3\varepsilon + 1)I_2(N-1) + 2(5\varepsilon + 2)I_2(N)& \\
- 2(\varepsilon - 2N + 1)I_3(N-1)& \\
=&- \frac{2(\varepsilon + 1)(3\varepsilon + 4)}{5\varepsilon + 6}B_1(N-1) + (5\varepsilon + 4)B_1(N) + \dots
\end{aligned}
$$

A coupled system of difference equations

$$
\begin{aligned}
&N I_1(N-1) - (\varepsilon + N + 1) I_1(N) + 2 I_2(N) \\
&\qquad = + \tfrac{4(N+2)}{3(N+1)}\varepsilon^{-3} + \Big(\tfrac{2(2N+1)}{3(N+1)} S_1(N) - \tfrac{2\left(6N^2+13N+8\right)}{3(N+1)^2}\Big)\varepsilon^{-2} + \dots
\end{aligned}
$$

$$
\begin{aligned}
&2(\varepsilon + 2N + 2) I_2(N) - 2(3\varepsilon + 2N + 1) I_2(N-1) \\
&\qquad + \varepsilon(3\varepsilon+2) I_1(N-1) - 2(\varepsilon+1) I_3(N-1) \\
&\qquad = \tfrac{8}{3}\varepsilon^{-3} + \Big(\tfrac{4}{3} S_1(N)^2 - \tfrac{4(N+1)}{N} S_1(N) + \tfrac{4}{3} S_2(N) + \zeta_2 + 6\Big)\varepsilon^{-1} + \dots
\end{aligned}
$$

$$
\begin{aligned}
&4(\varepsilon - N) I_3(N) - 2\varepsilon(3\varepsilon+2) I_1(N) + \varepsilon(3\varepsilon+2) I_1(N-1) \\
&\qquad - 2(3\varepsilon+1) I_2(N-1) + 2(5\varepsilon+2) I_2(N) \\
&\qquad - 2(\varepsilon - 2N + 1) I_3(N-1) \\
&\qquad = - \tfrac{8}{3}\varepsilon^{-3} - \Big(\tfrac{8}{3} S_1(N) - 4\Big)\varepsilon^{-2} \\
&\qquad\quad - \Big(\tfrac{4}{3} S_1(N)^2 - \tfrac{4(N+1)}{N} S_1(N) + \tfrac{4}{3} S_2(N) + \zeta_2 + 6\Big)\varepsilon^{-1} + \dots
\end{aligned}
$$

Step 2: Uncouple the system

$$\begin{aligned}
&\square I_1(N-1)+\square I_1(N)+\square I_2(N)\\
&\qquad=\square\varepsilon^{-3}+\square\varepsilon^{-2}+\square\varepsilon^{-1}+\dots\\
&\square I_2(N)+\square I_2(N-1)+\square I_1(N-1)+\square I_3(N-1)\\
&\qquad=\square\varepsilon^{-3}+\square\varepsilon^{-1}+\dots\\
&\square I_3(N)+\square I_1(N)+\square I_1(N-1)+\square I_2(N)+\square I_3(N-1)\\
&\qquad=\square\varepsilon^{-3}+\square\varepsilon^{-2}+\square\varepsilon^{-1}+\dots
\end{aligned}$$

Step 2: Uncouple the system

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&\Box I_2(N)+\Box I_2(N-1)+\Box I_1(N-1)+\Box I_3(N-1)\\
&\qquad=\Box\varepsilon^{-3}+\Box\varepsilon^{-1}+\dots\\
&\Box I_3(N)+\Box I_1(N)+\Box I_1(N-1)+\Box I_2(N)+\Box I_3(N-1)\\
&\qquad=\Box\varepsilon^{-3}+\Box\varepsilon^{-2}+\Box\varepsilon^{-1}+\dots
\end{aligned}$$

$\downarrow$ (uncoupling algorithms[a], S. Gerhold's `OrseSys.m`)

$$\begin{aligned}
&\Box I_1(N)+\Box I_1(N+1)+\Box I_1(N+2)+\Box I_1(N+3)\\
&\qquad=\Box\varepsilon^{-3}+\Box\varepsilon^{-2}+\Box\varepsilon^{-1}+\dots\\
&I_2(N)=\text{expression in } I_1(N)\\
&I_3(N)=\text{expression in } I_1(N)
\end{aligned}$$

[a] We use Zürcher's uncoupling algorithm (1994)

More precisely, we get:

$$
\begin{aligned}
&-2(N+1)(N+2)(\varepsilon+N+2)I_1(N)-(N+2)\big(2\varepsilon^2-5\varepsilon N-7\varepsilon-6N^2-28N-32\big)\\
&+\big(\varepsilon^3+4\varepsilon^2N+14\varepsilon^2-4\varepsilon N^2-13\varepsilon N-3\varepsilon-6N^3-50N^2-136N-120\big)I_1(N+2)\\
&-(\varepsilon-N-2)(\varepsilon+N+4)(\varepsilon+2N+8)I_1(N+3)\\
&=-\frac{4(N+2)}{3(N+3)}\varepsilon^{-3}+\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\varepsilon^{-2}+\dots
\end{aligned}
$$

Step 3: Solve the scalar recurrence

$$
\begin{aligned}
&-2(N+1)(N+2)(\varepsilon+N+2)I_1(N)-(N+2)\big(2\varepsilon^2-5\varepsilon N-7\varepsilon-6N^2-28N-32\big)\\
&+\big(\varepsilon^3+4\varepsilon^2N+14\varepsilon^2-4\varepsilon N^2-13\varepsilon N-3\varepsilon-6N^3-50N^2-136N-120\big)I_1(N+2)\\
&-(\varepsilon-N-2)(\varepsilon+N+4)(\varepsilon+2N+8)I_1(N+3)\\
&=-\frac{4(N+2)}{3(N+3)}\varepsilon^{-3}+\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\varepsilon^{-2}+\dots
\end{aligned}
$$

$$
\begin{aligned}
I_1(1) &= \tfrac{5}{\varepsilon^3}-\tfrac{163}{12\varepsilon^2}+\big(\tfrac{15\zeta_2}{8}+\tfrac{1223}{48}\big)\varepsilon^{-1}+\dots\\
I_1(2) &= \tfrac{130}{27\varepsilon^3}-\tfrac{695}{54\varepsilon^2}+\big(\tfrac{65\zeta_2}{36}+\tfrac{46379}{1944}\big)\varepsilon^{-1}+\dots\\
I_1(3) &= \tfrac{169}{36\varepsilon^3}-\tfrac{395}{32\varepsilon^2}+\big(\tfrac{169\zeta_2}{96}+\tfrac{470071}{20736}\big)\varepsilon^{-1}+\dots
\end{aligned}
$$

using, e.g., an extension of `MATAD` (M. Steinhauser) or tools given in [arXiv:1405.4259 [hep-ph]]

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$$
\begin{aligned}
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&+\big(\varepsilon^3+4\varepsilon^2N+14\varepsilon^2-4\varepsilon N^2-13\varepsilon N-3\varepsilon-6N^3-50N^2-136N-120\big)I_1(N+2)\\
&-(\varepsilon-N-2)(\varepsilon+N+4)(\varepsilon+2N+8)I_1(N+3)\\
&=-\frac{4(N+2)}{3(N+3)}\varepsilon^{-3}+\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\varepsilon^{-2}+\dots
\end{aligned}
$$

$$
\begin{aligned}
I_1(1) &= \frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+\Big(\frac{15\zeta_2}{8}+\frac{1223}{48}\Big)\varepsilon^{-1}+\dots\\
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I_1(3) &= \frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+\Big(\frac{169\zeta_2}{96}+\frac{470071}{20736}\Big)\varepsilon^{-1}+\dots
\end{aligned}
$$

using, e.g., an extension of `MATAD` (M. Steinhauser) or tools given in [arXiv:1405.4259 [hep-ph]]

$\downarrow$ (`Sigma.m`'s recurrence solver, see first slides)

$$
\begin{aligned}
I_1(N) &= \left(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1(N)}{3(N+1)}\right)\varepsilon^{-3}\\
&-\left(\frac{2\big(20N^3+58N^2+57N+22\big)}{3(N+1)^3}+\frac{S_1(N)^2}{N+1}+\frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2}-\frac{S_2(N)}{N+1}\right)\varepsilon^{-2}+\dots
\end{aligned}
$$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$, i.e.,

$$\begin{aligned} I_2(N) =& \square I_1(N) + \square I_1(N+1) + \square I_1(N+2) \\ & - \tfrac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\tfrac{6N^3+25N^2+33N+15}{3(N+1)^2(N+2)} + \tfrac{(-2N-1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots \end{aligned}$$

$$\begin{aligned} I_3(N) =& \square I_1(N) + \square I_1(N+1) + \square I_1(N+2) \\ & + \tfrac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\tfrac{-2N^3-3N^2+3N+3}{3(N+1)^2(N+2)} + \tfrac{(2N+1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots \end{aligned}$$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$, i.e.,

$$
\begin{aligned}
I_2(N) =& \Box I_1(N) + \Box I_1(N+1) + \Box I_1(N+2) \\
& - \tfrac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\tfrac{6N^3+25N^2+33N+15}{3(N+1)^2(N+2)} + \tfrac{(-2N-1)}{3(N+1)} S_1(N)\right)\varepsilon^{-2} + \dots \\
I_3(N) =& \Box I_1(N) + \Box I_1(N+1) + \Box I_1(N+2) \\
& + \tfrac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\tfrac{-2N^3-3N^2+3N+3}{3(N+1)^2(N+2)} + \tfrac{(2N+1)}{3(N+1)} S_1(N)\right)\varepsilon^{-2} + \dots
\end{aligned}
$$

This yields

$$
\begin{aligned}
I_2(N) =& \frac{4}{3\varepsilon^3} - \frac{2}{\varepsilon^2} + \left(-\tfrac{1}{3}S_1(N)^2 + \tfrac{2}{3}S_1(N) - \tfrac{1}{3}S_2(N) + \tfrac{5N+7}{3(N+1)} + \tfrac{\zeta_2}{2}\right)\varepsilon^{-1} + \dots \\
I_3(N) =& \tfrac{8}{3\varepsilon^3} + \left(\tfrac{4(N+2)}{3(N+1)}S_1(N) - \tfrac{4\left(4N^2+7N+2\right)}{3(N+1)^2}\right)\varepsilon^{-2} \\
& + \Bigg(-\tfrac{2\left(4N^2+11N+10\right)}{3(N+1)^2}S_1(N) + \tfrac{2\left(12N^3+32N^2+25N+2\right)}{3(N+1)^3} \\
& \quad + \tfrac{(N-2)}{3(N+1)}S_1(N)^2 + \tfrac{(N-2)}{3(N+1)}S_2(N) + \zeta_2\Bigg)\varepsilon^{-1} + \dots
\end{aligned}
$$

Summarizing:

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

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$$+$$

$$\begin{aligned} I_1(1) &= \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + \left(\frac{15\zeta_2}{8} + \frac{1223}{48}\right)\varepsilon^{-1} + \dots \\ I_1(2) &= \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + \left(\frac{65\zeta_2}{36} + \frac{46379}{1944}\right)\varepsilon^{-1} + \dots \\ I_1(3) &= \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + \left(\frac{169\zeta_2}{96} + \frac{470071}{20736}\right)\varepsilon^{-1} + \dots \end{aligned}$$

Summarizing:

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

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$$\downarrow$$

$$\begin{aligned} I_1(N) =& \left(\frac{4\left(3N^2+6N+4\right)}{3(N+1)^2} + \frac{4S_1(N)}{3(N+1)}\right)\varepsilon^{-3} \\ & - \left(\frac{2\left(20N^3+58N^2+57N+22\right)}{3(N+1)^3} + \frac{S_1(N)^2}{N+1} + \frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2} - \frac{S_2(N)}{N+1}\right)\varepsilon^{-2} + \ldots \\ I_2(N) =& \frac{4}{3\varepsilon^3} - \frac{2}{\varepsilon^2} + \left(-\frac{1}{3}S_1(N)^2 + \frac{2}{3}S_1(N) - \frac{1}{3}S_2(N) + \frac{5N+7}{3(N+1)} + \frac{\zeta_2}{2}\right)\varepsilon^{-1} + \ldots \\ I_3(N) =& \frac{8}{3\varepsilon^3} + \left(\frac{4(N+2)}{3(N+1)}S_1(N) - \frac{4\left(4N^2+7N+2\right)}{3(N+1)^2}\right)\varepsilon^{-2} \\ & + \left(-\frac{2\left(4N^2+11N+10\right)}{3(N+1)^2}S_1(N) + \frac{2\left(12N^3+32N^2+25N+2\right)}{3(N+1)^3} + \ldots \right. \end{aligned}$$

The full algorithm

Given a coupled system

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix} = A \overbrace{\begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix}}^{\text{unknown}} + \begin{pmatrix} \hat{r}_1(x) \\ \hat{r}_2(x) \\ \dots \\ \hat{r}_n(x) \end{pmatrix}$$

The full algorithm

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A: a given invertible $n \times n$ matrix with elements form $\mathbb{Q}(N, \varepsilon)$.

The full algorithm

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$\hat{r}_i(x)$: given as power series in x whose coefficients in terms of indefinite nested sums and products are high enough expanded in ε

The full algorithm

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where $\hat{I}_i(x)$ has a power series representation $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)$

The full algorithm

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$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix} = A \overbrace{\begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix}}^{\text{unknown}} + \begin{pmatrix} \hat{r}_1(x) \\ \hat{r}_2(x) \\ \dots \\ \hat{r}_n(x) \end{pmatrix}$$

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$+$

Given the initial values of $I_1(N)$ expanded high enough in ε.

The full algorithm

Given a coupled system

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix} = A \overbrace{\begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix}}^{\text{unknown}} + \begin{pmatrix} \hat{r}_1(x) \\ \hat{r}_2(x) \\ \dots \\ \hat{r}_n(x) \end{pmatrix}$$

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$+$

Given the initial values of $I_1(N)$ expanded high enough in ε.

$\downarrow$

A decision algorithm which **computes** (if possible) the ε-expansions of $F_1(N), \dots, F_n(N)$ in terms of indefinite nested sums and products; **special case:** harmonic sums, S-sums or nested binomial sums.

The full algorithm

Given a coupled system

can be determined as a preprocessing step

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix} = A \underbrace{\begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \dots \\ \hat{I}_n(x) \end{pmatrix}}_{\text{unknown}} + \begin{pmatrix} \hat{r}_1(x) \\ \hat{r}_2(x) \\ \dots \\ \hat{r}_n(x) \end{pmatrix}$$

A: a given invertible $n \times n$ matrix with elements form $\mathbb{Q}(N, \varepsilon)$.

$\hat{r}_i(x)$: given as power series in x whose coefficients in terms of indefinite nested sums and products are high enough expanded in ε

where $\hat{I}_i(x)$ has a power series representation $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N)$

$+$

Given the initial values of $I_1(N)$ expanded high enough in ε.

$\downarrow$

A decision algorithm which **computes** (if possible) the ε-expansions of $F_1(N), \dots, F_n(N)$ in terms of indefinite nested sums and products; **special case:** harmonic sums, S-sums or nested binomial sums.

Goal: Expand the 92 master integrals

$$\left.\begin{array}{l} B_1(N) \\ B_2(N) \\ \vdots \\ B_{54}(N) \end{array}\right\}$$ 54 by symbolic summation

$$\left.\begin{array}{l} B_{55}(N) \\ B_{56}(N) \\ \vdots \\ B_{62}(N) \end{array}\right\}$$ 8 by symbolic integration and recurrence solving

$$\left.\begin{array}{l} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \vdots \\ \hat{I}_{32}(x) \end{array}\right\}$$ `REDUZE_2` (A.v. Manteuffel) produces a (recursively organized) coupled differential system

I_4, I_5, I_6, I_7: determined by the most complicated system

I_4, I_5, I_6, I_7: determined by the most complicated system

We uncouple it in $I_4(N)$ and get the linear recurrence

$$a_0(N,\varepsilon)I_4(N) + a_1(N,\varepsilon)I_4(N+1) + \dots + a_5(N,\varepsilon)I_4(N+5)$$
$$= h_{-1}(N)\varepsilon^{-1} + h_0(N)\varepsilon^0 + \dots + h_4(N)\varepsilon^4 + \dots$$

$a_i(N,\varepsilon)$: large polynomials in ε and N

$h_i(N)$: given in terms of 726 S-sums up to weight≤ 7
(our symbolic integration problem occurs on the right hand side)

Solving this recurrence with the given initial values yields the ε-expansion

$$I_4(N) = \varepsilon^{-1}F_{-1}(N) + \varepsilon^0 F_0(N) + \dots + \varepsilon^4 F_4(N) + \dots$$

Solving this recurrence with the given initial values yields the ε-expansion

$$I_4(N) = \varepsilon^{-1}\boxed{F_{-1}(N)} + \varepsilon^0 F_0(N) + \dots + \varepsilon^4 F_4(N) + \dots$$

with

$$F_{-1}(N) = \big((-1)^N - 2\big)\frac{2S_2(N)}{(N+1)^2} + \big((-1)^N - 1\big)\frac{4S_{-2}(N)}{(N+1)^2} - \frac{8}{(N+1)^4} + \frac{6(-1)^N}{(N+1)^4}$$

Solving this recurrence with the given initial values yields the ε-expansion

$$I_4(N) = \varepsilon^{-1} F_{-1}(N) + \varepsilon^0 \boxed{F_0(N)} + \dots + \varepsilon^4 F_4(N) + \dots$$

with $F_0(N)$ given in terms of the (generalized) harmonic sums

$$S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_{-2,1}(N), S_{2,1}(N),$$
$$S_1\left(\tfrac{1}{2}, N\right), S_3\left(-\tfrac{1}{2}, N\right), S_3\left(\tfrac{1}{2}, N\right), S_{2,1}\left(-1, \tfrac{1}{2}, N\right), S_{2,1}\left(1, \tfrac{1}{2}, N\right).$$

plus 19 (inverse) binomial sums, like

$$\sum_{k=1}^{N} (-2)^k k^2 \binom{2k}{k} \sum_{j=1}^{k} \frac{2^{-j} \sum_{i=1}^{j} \frac{(-1)^i}{i^2}}{j}.$$

J. Ablinger, J. Blümlein, C. Raab, CS, 2015

Solving this recurrence with the given initial values yields the ε-expansion

$$I_4(N) = \boxed{F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^0 + \dots + F_4(N)\varepsilon^{-4}} + \dots$$

Summary for $F_{-1}(N), \dots, F_4(N)$:

- Total calculation time: $3\frac{1}{2}$ days

Solving this recurrence with the given initial values yields the ε-expansion

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Summary for $F_{-1}(N), \ldots, F_4(N)$:

- Total calculation time: $3\frac{1}{2}$ days
- 928 (generalized) harmonic sums up to weight 8

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Summary for $F_{-1}(N), \dots, F_4(N)$:

- Total calculation time: $3\frac{1}{2}$ days
- 928 (generalized) harmonic sums up to weight 8
- 2598 binomial sums up to nesting depth 7, like e.g.,

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=1}^{j} \frac{1}{\binom{2i}{i} i^2} \sum_{i_4=1}^{i} (-2)^{i_4} \binom{2i_4}{i_4} i_4^2 \sum_{r=1}^{i_4} \frac{2^{-r}}{r} \sum_{s=1}^{r} \frac{(-1)^s}{s^2},$$

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=1}^{j} \frac{1}{i} \sum_{i_4=1}^{i} \frac{1}{i_4} \sum_{r=1}^{i_4} (-2)^r \binom{2r}{r} r^2 \sum_{s=1}^{r} \frac{2^{-s}}{s} \sum_{i_7=1}^{s} \frac{1}{i_7^2}.$$

All arising sums are algebraically independent!

Solving this recurrence with the given initial values yields the ε-expansion

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Summary for $F_{-1}(N), \dots, F_4(N)$:

- Total calculation time: $3\frac{1}{2}$ days
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$$\sum_{k=1}^{N}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{\binom{2i}{i}i^2}\sum_{i_4=1}^{i}(-2)^{i_4}\binom{2i_4}{i_4}i_4^2\sum_{r=1}^{i_4}\frac{2^{-r}}{r}\sum_{s=1}^{r}\frac{(-1)^s}{s^2},$$

$$\sum_{k=1}^{N}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{i}\sum_{i_4=1}^{i}\frac{1}{i_4}\sum_{r=1}^{i_4}(-2)^r\binom{2r}{r}r^2\sum_{s=1}^{r}\frac{2^{-s}}{s}\sum_{i_7=1}^{s}\frac{1}{i_7^2}.$$

  All arising sums are algebraically independent!
- Possible by combining difference ring techniques with quasi-shuffle algebras (using Sigma and HarmonicSums)

Goal: Expand the 92 master integrals

$$\left.\begin{array}{l} B_1(N)\\ B_2(N)\\ \vdots\\ B_{54}(N)\end{array}\right\}$$ 54 by symbolic summation

$$\left.\begin{array}{l} B_{55}(N)\\ B_{56}(N)\\ \vdots\\ B_{62}(N)\end{array}\right\}$$ 8 by symbolic integration and recurrence solving

$$\left.\begin{array}{l} B_{63}(N)\\ B_{64}(N)\\ \vdots\\ B_{92}(N)\end{array}\right\}$$ 32 by solving coupled (recursively organized) differential systems

$$\sum_{N=0}^{\infty} D_{12}(N)x^N = e_1(x,\varepsilon)\hat{B}_1(x) + e_2(x,\varepsilon)\hat{B}_2(x) + \dots$$
$$+ e_i(x,\varepsilon)\hat{B}_i(x) + \dots e_{92}(x,\varepsilon)\hat{B}_{92}(x)$$

Note: the $\hat{B}_i(x)$ can be represented as power series

$$\hat{B}_i(x) = \sum_{N=0}^{\infty} B_i(N)x^N$$

Goal: Expand the 92 master integrals

$$B_i(N) = \overbrace{b_{-3}(N)\varepsilon^{-3} + b_{-2}(N)\varepsilon^{-2} + b_{-1}(N)\varepsilon^{-1} + b_0(N)\varepsilon^0 + \dots}$$

$$\sum_{N=0}^{\infty} D_{12}(N)x^N = e_1(x,\varepsilon)\hat{B}_1(x) + e_2(x,\varepsilon)\hat{B}_2(x) + \dots$$

$$+ e_i(x,\varepsilon)\hat{B}_i(x) + \dots e_{92}(x,\varepsilon)\hat{B}_{92}(x)\Big)$$

Note: the $\hat{B}_i(x)$ can be represented as power series

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Goal: Expand the 92 master integrals

$$B_i(N) = \overbrace{b_{-3}(N)\varepsilon^{-3} + b_{-2}(N)\varepsilon^{-2} + b_{-1}(N)\varepsilon^{-1} + b_0(N)\varepsilon^0 + \dots}$$

Take the coefficient of x^N

further technologies are provided in `SumProduction.m`

$$D_{12}(N) = [x^N]\Big(e_1(x,\varepsilon)\hat{B}_1(x) + e_2(x,\varepsilon)\hat{B}_2(x) + \dots + e_i(x,\varepsilon)\hat{B}_i(x) + \dots e_{92}(x,\varepsilon)\hat{B}_{92}(x)\Big)$$

Note: the $\hat{B}_i(x)$ can be represented as power series

$$\hat{B}_i(x) = \overbrace{\sum_{N=0}^{\infty} B_i(N)x^N}$$

Goal: Expand the 92 master integrals

$$B_i(N) = \overbrace{b_{-3}(N)\varepsilon^{-3} + b_{-2}(N)\varepsilon^{-2} + b_{-1}(N)\varepsilon^{-1} + b_0(N)\varepsilon^0 + \dots}$$

$$D_{12}(N) = \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{128(N^2+N+1)}{3N(N+1)^2(N+2)} - \frac{64S_1^2}{3(N+1)(N+2)} + \frac{128(N+3)S_1}{3(N+1)^2(N+2)} - \frac{256S_{-2}}{3(N+1)(N+2)} + (-1)^N\frac{128}{3N(N+1)^2(N+2)} - \frac{64S_2}{(N+1)(N+2)}\Bigg]\varepsilon^{-3}$$

$$D_{12}(N) = \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{128(N^2+N+1)}{3N(N+1)^2(N+2)} - \frac{64S_1^2}{3(N+1)(N+2)}$$
$$+\frac{128(N+3)S_1}{3(N+1)^2(N+2)} - \frac{256S_{-2}}{3(N+1)(N+2)} + (-1)^N\frac{128}{3N(N+1)^2(N+2)} - \frac{64S_2}{(N+1)(N+2)}\Bigg]\varepsilon^{-3}$$
$$+\left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{64P_2}{3N(N+1)^3(N+2)^2} + \frac{32S_1^3}{3(N+1)(N+2)}$$
$$-\frac{32(2N^3+20N^2+35N+12)S_1^2}{3N(N+1)^2(N+2)^2} - \frac{32(2N+1)(4N^3+10N^2+17N+20)S_2}{3N(N+1)^2(N+2)^2}$$
$$+\left(\frac{64P_3}{3N(N+1)^3(N+2)^2} - \frac{96S_2}{(N+1)(N+2)}\right)S_1 - \frac{256(2N+5)S_3}{3(N+1)(N+2)}$$
$$+\left(-\frac{128P_1}{3N(N+1)^2(N+2)^2} - \frac{512S_1}{3(N+1)(N+2)}\right)S_{-2} + \frac{256(N+4)S_{2,1}}{3(N+1)(N+2)}$$
$$+\frac{512S_{-2,1}}{3(N+1)(N+2)} + (-1)^N\frac{64(4N^3+16N^2+28N+21)}{3N(N+1)^3(N+2)^2}\Bigg]\varepsilon^{-2}$$

$$D_{12}(N) = \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{128(N^2+N+1)}{3N(N+1)^2(N+2)} - \frac{64S_1^2}{3(N+1)(N+2)} + \frac{128(N+3)S_1}{3(N+1)^2(N+2)} - \frac{256S_{-2}}{3(N+1)(N+2)} + (-1)^N\frac{128}{3N(N+1)^2(N+2)} - \frac{64S_2}{(N+1)(N+2)}\Bigg]\varepsilon^{-3}$$

$$+ \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{64P_2}{3N(N+1)^3(N+2)^2} + \frac{32S_1^3}{3(N+1)(N+2)} - \frac{32(2N^3+20N^2+35N+12)S_1^2}{3N(N+1)^2(N+2)^2} - \frac{32(2N+1)(4N^3+10N^2+17N+20)S_2}{3N(N+1)^2(N+2)^2} + \left(\frac{64P_3}{3N(N+1)^3(N+2)^2} - \frac{96S_2}{(N+1)(N+2)}\right)S_1 - \frac{256(2N+5)S_3}{3(N+1)(N+2)} + \left(-\frac{128P_1}{3N(N+1)^2(N+2)^2} - \frac{512S_1}{3(N+1)(N+2)}\right)S_{-2} + \frac{256(N+4)S_{2,1}}{3(N+1)(N+2)} + \frac{512S_{-2,1}}{3(N+1)(N+2)} + (-1)^N\frac{64(4N^3+16N^2+28N+21)}{3N(N+1)^3(N+2)^2}\Bigg]\varepsilon^{-2}$$

$$+(\dots)\varepsilon^{-1}$$

Arising objects(harmonic sums):

$$\zeta_2, (-1)^N, S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N),$$
$$S_{-3,1}(N), S_{-2,1}(N), S_{-2,2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

[J.A.M. Vermaseren, 1998; J. Blümlein/S. Kurth, 1998]

$$D_{12}(N) = \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{128(N^2+N+1)}{3N(N+1)^2(N+2)} - \frac{64S_1^2}{3(N+1)(N+2)} + \frac{128(N+3)S_1}{3(N+1)^2(N+2)} - \frac{256S_{-2}}{3(N+1)(N+2)} + (-1)^N\frac{128}{3N(N+1)^2(N+2)} - \frac{64S_2}{(N+1)(N+2)}\Bigg]\varepsilon^{-3}$$

$$+\left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{64P_2}{3N(N+1)^3(N+2)^2} + \frac{32S_1^3}{3(N+1)(N+2)} - \frac{32(2N^3+20N^2+35N+12)S_1^2}{3N(N+1)^2(N+2)^2} - \frac{32(2N+1)(4N^3+10N^2+17N+20)S_2}{3N(N+1)^2(N+2)^2} + \left(\frac{64P_3}{3N(N+1)^3(N+2)^2} - \frac{96S_2}{(N+1)(N+2)}\right)S_1 - \frac{256(2N+5)S_3}{3(N+1)(N+2)} + \left(-\frac{128P_1}{3N(N+1)^2(N+2)^2} - \frac{512S_1}{3(N+1)(N+2)}\right)S_{-2} + \frac{256(N+4)S_{2,1}}{3(N+1)(N+2)} + \frac{512S_{-2,1}}{3(N+1)(N+2)} + (-1)^N\frac{64(4N^3+16N^2+28N+21)}{3N(N+1)^3(N+2)^2}\Bigg]\varepsilon^{-2}$$

$$+(\dots)\varepsilon^{-1} + (\dots)\varepsilon^{0}$$

Arising objects(harmonic sums):

$S_{-5}(N), S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_5(N), S_{-4,1}(N), S_{-3,1}(N),$
$S_{-2,-3}(N), S_{-2,1}(N), S_{-2,2}(N), S_{-2,3}(N), S_1(-2,N), S_{2,-3}(N), S_{2,1}(N), S_{2,3}(N), S_{3,1}(N),$
$S_{4,1}(N), S_{-3,1,1}(N), S_{-2,1,-2}(N), S_{-2,1,1}(N), S_{-2,2,1}(N), S_{2,1,-2}(N), S_{2,1,1}(N),$
$S_{2,2,1}(N), S_{3,1,1}(N), S_{-2,1,1,1}(N), S_{2,1,1,1}(N)$

[J.A.M. Vermaseren, 1998; J. Blümlein/S. Kurth, 1998]

$$
\begin{aligned}
D_{12}(N) &= \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{128(N^2+N+1)}{3N(N+1)^2(N+2)} - \frac{64S_1^2}{3(N+1)(N+2)} \\
&\quad + \frac{128(N+3)S_1}{3(N+1)^2(N+2)} - \frac{256S_{-2}}{3(N+1)(N+2)} + (-1)^N\frac{128}{3N(N+1)^2(N+2)} - \frac{64S_2}{(N+1)(N+2)}\Bigg]\varepsilon^{-3} \\
&+ \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{64P_2}{3N(N+1)^3(N+2)^2} + \frac{32S_1^3}{3(N+1)(N+2)} \\
&\quad - \frac{32\left(2N^3+20N^2+35N+12\right)S_1^2}{3N(N+1)^2(N+2)^2} - \frac{32(2N+1)\left(4N^3+10N^2+17N+20\right)S_2}{3N(N+1)^2(N+2)^2} \\
&\quad + \left(\frac{64P_3}{3N(N+1)^3(N+2)^2} - \frac{96S_2}{(N+1)(N+2)}\right)S_1 - \frac{256(2N+5)S_3}{3(N+1)(N+2)} \\
&\quad + \left(-\frac{128P_1}{3N(N+1)^2(N+2)^2} - \frac{512S_1}{3(N+1)(N+2)}\right)S_{-2} + \frac{256(N+4)S_{2,1}}{3(N+1)(N+2)} \\
&\quad + \frac{512S_{-2,1}}{3(N+1)(N+2)} + (-1)^N\frac{64\left(4N^3+16N^2+28N+21\right)}{3N(N+1)^3(N+2)^2}\Bigg]\varepsilon^{-2} \\
&+(\ldots)\varepsilon^{-1}+(\ldots)\varepsilon^{0}
\end{aligned}
$$

Arising objects(generalized harmonic sums):

$$
S_{2,1,2}\left(-2,\frac{1}{2},1,N\right) = \sum_{j=1}^{N}\frac{(-2)^j\displaystyle\sum_{i=1}^{j}\frac{2^{-i}\displaystyle\sum_{k=1}^{i}\frac{1}{k^2}}{i}}{j^2}
$$

S. Moch, P. Uwer, S. Weinzierl, 2002

$$D_{12}(N) = \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{128(N^2+N+1)}{3N(N+1)^2(N+2)} - \frac{64S_1^2}{3(N+1)(N+2)}$$
$$+\frac{128(N+3)S_1}{3(N+1)^2(N+2)} - \frac{256S_{-2}}{3(N+1)(N+2)} + (-1)^N\frac{128}{3N(N+1)^2(N+2)} - \frac{64S_2}{(N+1)(N+2)}\Bigg]\varepsilon^{-3}$$
$$+\left(\frac{C_A}{2} - C_F\right)(C_A - C_F)T_F\Bigg[-\frac{64P_2}{3N(N+1)^3(N+2)^2} + \frac{32S_1^3}{3(N+1)(N+2)}$$
$$-\frac{32(2N^3+20N^2+35N+12)S_1^2}{3N(N+1)^2(N+2)^2} - \frac{32(2N+1)(4N^3+10N^2+17N+20)S_2}{3N(N+1)^2(N+2)^2}$$
$$+\left(\frac{64P_3}{3N(N+1)^3(N+2)^2} - \frac{96S_2}{(N+1)(N+2)}\right)S_1 - \frac{256(2N+5)S_3}{3(N+1)(N+2)}$$
$$+\left(-\frac{128P_1}{3N(N+1)^2(N+2)^2} - \frac{512S_1}{3(N+1)(N+2)}\right)S_{-2} + \frac{256(N+4)S_{2,1}}{3(N+1)(N+2)}$$
$$+\frac{512S_{-2,1}}{3(N+1)(N+2)} + (-1)^N\frac{64(4N^3+16N^2+28N+21)}{3N(N+1)^3(N+2)^2}\Bigg]\varepsilon^{-2}$$
$$+(\dots)\varepsilon^{-1}+(\dots)\varepsilon^{0}$$

Arising objects(nested binomial sums):

$$\sum_{i=1}^{N}(-2)^i\binom{2i}{i}\sum_{k=1}^{i}\frac{1}{k\binom{2k}{k}}S_{1,2}\left(\frac{1}{2},1,i\right) \qquad \sum_{i=1}^{N}\frac{\sum_{k=1}^{i}\frac{(-1)^k\binom{2k}{k}S_2(k)}{k}}{(1+i)\binom{2i}{i}}$$

J. Ablinger, J. Blümlein, C. Raab, CS, 2015

New algorithms for asymptotic expansions

using the underlying integral representation (available in HarmonicSums.m)

- Generalized harmonic sums

$$
\begin{aligned}
&S_{1,1,1,1}\left(2,\tfrac{1}{2},1,1,N\right) = \\
&\quad= \frac{-21\zeta_2^2}{20} + \frac{1}{N} + \frac{1}{8N^2} + \frac{295}{216N^3} - \frac{1115}{96N^4} + O(N^{-5}) \\
&\qquad + \Big(\frac{1}{2N} - \frac{3}{4N^2} + \frac{19}{12N^3} - \frac{5}{N^4} + O(N^{-5})\Big)\zeta_2 \\
&\qquad + 2^N\Big(\frac{3}{2N} + \frac{3}{2N^2} + \frac{9}{2N^3} + \frac{39}{2N^4} + O(N^{-5})\Big)\zeta_3 \\
&\qquad + \Big(\frac{1}{N} + \frac{3}{4N^2} - \frac{157}{36N^3} + \frac{19}{N^4} + O(N^{-5})\Big)(\log(N) + \gamma) \\
&\qquad + \Big(\frac{1}{2N} - \frac{3}{4N^2} + \frac{19}{12N^3} - \frac{5}{N^4} + O(N^{-5})\Big)(\log(N) + \gamma)^2
\end{aligned}
$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

New algorithms for asymptotic expansions

using the underlying integral representation (available in HarmonicSums.m)

- Cyclotomic harmonic sums

$$\sum_{k=1}^{N} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \left(-3+\frac{35\zeta_3}{16}\right)\zeta_2 - \frac{31\zeta_5}{8}$$
$$+\frac{1}{N} - \frac{33}{32N^2} + \frac{17}{16N^3} - \frac{4795}{4608N^4} + O(N^{-5})$$
$$+\log(2)\Big(6\zeta_2 - \frac{1}{N} + \frac{9}{8N^2} - \frac{7}{6N^3} + \frac{209}{192N^4} + O(N^{-5})\Big)$$
$$+\Big(-\frac{7}{4} - \frac{7}{16N} + \frac{7}{16N^2} - \frac{77}{192N^3} + \frac{21}{64N^4} + O(N^{-5})\Big)\zeta_3$$
$$+\Big(\frac{1}{16N^2} - \frac{1}{8N^3} + \frac{65}{384N^4} + O(N^{-5})\Big)(\log(N)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

New algorithms for asymptotic expansions

using the underlying integral representation (available in HarmonicSums.m)

- Nested binomial sums

$$\sum_{j=1}^{N}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2} = 7\zeta_3+\sqrt{\pi}\sqrt{N}\Bigg\{\Bigg[-\frac{2}{N}+\frac{5}{12N^2}-\frac{21}{320N^3}-\frac{223}{10752N^4}+\frac{671}{49152N^5} + \frac{11635}{1441792N^6}-\frac{1196757}{136314880N^7}-\frac{376193}{50331648N^8}+\frac{201980317}{18253611008N^9} + O(N^{-10})\Bigg]\ln(\bar{N})-\frac{4}{N}+\frac{5}{18N^2}-\frac{263}{2400N^3}+\frac{579}{12544N^4}+\frac{10123}{1105920N^5} - \frac{1705445}{71368704N^6}-\frac{27135463}{11164188672N^7}+\frac{197432563}{7927234560N^8}+\frac{405757489}{775778467840N^9} + O(N^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, 2014. arXiv:1407.1822 [hep-th]

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