# One-Mass Two-Loop Master Integrals for Mixed $\alpha_s$-Electroweak Drell-Yan Production

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work ongoing with Andreas von Manteuffel

The PRISMA Cluster of Excellence
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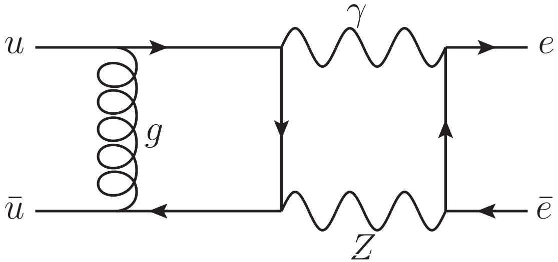
# Outline

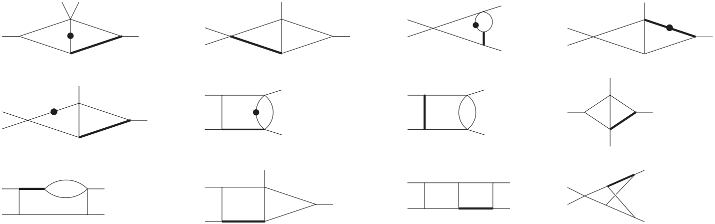
# Why Drell-Yan?




- PDF determination/tests of PDF universality
- Precision $Z$ boson physics
- Discovery of new particles!

# Overview

- Massless two-loop QCD $\times$ QED virtual corrections known for some time (W. Kilgore and C. Sturm, Phys. Rev. **D85**, 033005, 2012).
- Very recently, approximate results have been obtained in the resonance region (S. Dittmaier *et. al.*, Nucl. Phys. **B885**, 318, 2014).
- We have taken first steps towards an exact determination of the SM gauge boson mass dependence by computing all one-mass master integrals which enter into the virtual corrections.



# Integration By Parts in $d$ Dimensions

F. Tkachov, Phys. Lett. **B100**, 65, 1981; K. Chetyrkin and F. Tkachov, Nucl. Phys. **B192**, 159, 1981

$$\begin{aligned} 0 &= \int \frac{d^d\ell}{(2\pi)^d} \frac{\partial}{\partial \ell_\mu} \left( \frac{\ell_\mu}{(\ell^2 - m^2)^a} \right) \\ &= \int \frac{d^d\ell}{(2\pi)^d} \left( \frac{d}{(\ell^2 - m^2)^a} - \frac{2a\ell^2}{(\ell^2 - m^2)^{a+1}} \right) \\ &= (d-2a) I(a) - 2am^2 I(a+1) \end{aligned}$$

$$\Longrightarrow I(a) = \frac{(-1)^a \Gamma(a - d/2)}{\Gamma(1-d/2)\Gamma(a)\left(m^2\right)^{a-1}} I(1)$$

In general, one must consider all integration by parts relations generated by $\{\ell_1^\mu, \ldots, \ell_L^\mu\}$ AND $\{k_1^\mu, \ldots, k_N^\mu\}$ for each differentiation variable $\ell_j^\mu$ and deal with irreducible numerators.

# Can We Solve These IBP Relations?

S. Laporta, Int. J. Mod. Phys. **A15**, 5087, 2000

Suppose we want to solve the system of IBP recurrence relations to determine the master integrals for a given multi-loop topology:

- For most interesting examples a highly non-trivial system of recurrence relations results.
- Recurrence relations are typically hard to solve directly.
- The so-called Laporta algorithm maps the problem to a large linear system which can be solved using linear algebra.

Widely-used public implementations of Laporta's algorithm exist (principally `FIRE 5` and `Reduze 2`).

A. V. Smirnov, Comput. Phys. Commun. **189**, 182, 2014

A. von Manteuffel and C. Studerus, arXiv:1201.4330

# The Method Of Differential Equations

- The method of differential equations for multi-loop Feynman integrals (E. Remiddi, Nuovo Cim. **A110**, 1435, 1997; T. Gehrmann and E. Remiddi, Comput. Phys. Commun. **141**, 296, 2001) involves first deriving a system of first-order differential equations by differentiating the integrals of interest with respect to the available parameters and then using integration by parts identities to rewrite the derivatives obtained in terms of master integrals.
- The system of differential equations obtained can be solved order-by-order in $\epsilon$ up to constants. In practice, a large percentage of the master integrals are actually completely determined in this approach because many of the integration constants are completely determined by the physics.
- Unfortunately, the method is cumbersome to apply because an order-by-order solution is complicated by the fact that the systems obtained are typically coupled in a non-trivial way.

# Normal Form Systems

- Recently, Henn (Phys. Rev. Lett. **110**, 251601, 2013) suggested a novel approach to the decoupling of first-order systems of differential equations for Feynman integrals.
  (see also A. V. Kotikov, "Subtleties in Quantum Field Theory," 150)
- When the method applies, it provides a clean prescription for the computation which is transparent and in many cases usable even by non-experts to obtain results to arbitrarily high orders in $\epsilon$.
- Proceed by finding a basis of integrals $\mathbf{f}(\epsilon, x, y) = \{f_1(\epsilon, x, y), \ldots, f_{49}(\epsilon, x, y)\}$ with $\epsilon$ expansions of the form $f_i(\epsilon, x, y) = \sum_{n=0}^{\infty} c_i^{(n)}(x, y)\epsilon^n$ such that:

$$\mathbf{I}(\epsilon, x, y) = \underset{\sim}{\mathbf{B}}(\epsilon, x, y)\mathbf{f}(\epsilon, x, y) \qquad \Longrightarrow$$

$$\frac{\partial}{\partial x}\mathbf{I}(\epsilon, x, y) = \underset{\sim}{\mathbf{S}}_x(\epsilon, x, y)\mathbf{I}(\epsilon, x, y) \longrightarrow \frac{\partial}{\partial x}\mathbf{f}(\epsilon, x, y) = \epsilon \underset{\sim}{\mathbf{A}}_x(x, y)\mathbf{f}(\epsilon, x, y)$$

# What Is Special About A Normal Form?

Here, one obtains an PDE such that the functional form of the term of $\mathcal{O}\left(\epsilon^{n+1}\right)$ is completely determined by the term of $\mathcal{O}\left(\epsilon^{n}\right)$:

$$\frac{\partial}{\partial x}\mathbf{c}^{(n+1)}(x,y) = \underset{\sim}{\mathbf{A}}_x(x,y)\mathbf{c}^{(n)}(x,y)$$

Here, the elements of $\underset{\sim}{\mathbf{A}}_x(x,y)$ are linear combinations of weights drawn from the set {x, x+1, y, y-1, y+1, y-x, x+y+x y}.

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For the problem at hand, a number of different techniques are actually used to fix the integration constants. For example, we perform explicit integrations, look at asymptotic limits of normal form integrals, and exploit unitarity (more on this later).

# Unlock The Full Power Of The Normal Form

We have the partial differential equations

$$\begin{aligned}\frac{\partial}{\partial x}\mathbf{c}^{(n+1)}(x,y) &= \underset{\sim}{\mathbf{A}}_x(x,y)\mathbf{c}^{(n)}(x,y)\\ \frac{\partial}{\partial y}\mathbf{c}^{(n+1)}(x,y) &= \underset{\sim}{\mathbf{A}}_y(x,y)\mathbf{c}^{(n)}(x,y)\end{aligned}$$

for $x = \frac{-t}{-s}$ and $y = \frac{m^2}{-s}$. These equations can be rewritten as a relation between total differentials:

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C. Duhr, JHEP **1208**, 043, 2012; L. J. Dixon *et. al.*, JHEP **1406**, 116, 2014

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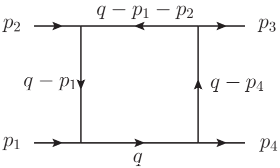
In quite general situations, a Duhr-Gangl-Rhodes approach (see JHEP **1210**, 075, 2012) can then be successfully applied to generate an ansatz for $\mathbf{c}^{(n+1)}(x,y)$ in terms of unknown integration constants.

# Example: One-Loop Massless Box Integral Family

Z. Bern *et. al.*, Nucl. Phys. **B412**, 751, 1994




$$I_2(\epsilon) = \frac{(-s)^{1+\epsilon}\epsilon e^{\epsilon\gamma_E}}{i\pi^{2-\epsilon}} \int \frac{\mathrm{d}^d q}{q^2(q-p_1-p_2)^2} = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)e^{\epsilon\gamma_E}}{\Gamma(1-2\epsilon)}$$

$$I_2(\epsilon,s,t) = \frac{(-s)^{1+\epsilon}\epsilon e^{\epsilon\gamma_E}}{i\pi^{2-\epsilon}} \int \frac{\mathrm{d}^d q}{(q-p_1)^2(q-p_4)^2} = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)e^{\epsilon\gamma_E}\left(\frac{-t}{-s}\right)^{-\epsilon}}{\Gamma(1-2\epsilon)}$$

$$I_4(\epsilon,s,t) = \frac{(-s)^{1+\epsilon}(-t)\epsilon^2 e^{\epsilon\gamma_E}}{i\pi^{2-\epsilon}} \int \frac{\mathrm{d}^d q}{q^2(q-p_1)^2(q-p_1-p_2)^2(q-p_4)^2}$$

# The Crucial Relation

In differential form, the ODE for the box function reads:

$$\begin{aligned} \mathrm{d}I_4(\epsilon,x) &= \epsilon\Big(-2\mathrm{d}\ln(1+x)I_2(\epsilon)+\Big(2\mathrm{d}\ln(x)-2\mathrm{d}\ln(1+x)\Big)I_2(\epsilon,x) \\ &\quad +\Big(\mathrm{d}\ln(1+x)-\mathrm{d}\ln(x)\Big)I_4(\epsilon,x)\Big) \end{aligned}$$

We Taylor expand the integrals in terms of functions of $x$ of uniform weight, *e.g.* $I_4(\epsilon,x)=\sum_{i=0}^{\infty} I_4^{(i)}(x)\epsilon^i$. A consequence of the coproduct construction and the above total differential is that

$$\begin{aligned} \Delta_{i-1,1}\left(I_4^{(i)}(x)\right) &= I_2^{(i-1)}\otimes\Big(-2\ln(1+x)\Big) \\ &\quad +I_2^{(i-1)}(x)\otimes\Big(-2\ln(1+x)+2\ln(x)\Big) \\ &\quad +I_4^{(i-1)}(x)\otimes\Big(\ln(1+x)-\ln(x)\Big) \end{aligned}$$

# The Crucial Relation Continued

If we let $a_i$, $b_i$, and $c_i$ denote the integration constants of $I_2(\epsilon)$, $I_2(\epsilon, x)$, and $I_4(\epsilon, x)$ of weight $i$, then we see that

$$I_4^{(1)}(x) = c_1 + a_0\Big(-2\ln(1+x)\Big) + b_0\Big(-2\ln(1+x) + 2\ln(x)\Big) + c_0\Big(\ln(1+x) - \ln(x)\Big)$$

Thus begins the bootstrap:

$$\begin{aligned}\Delta_{1,1}\left(I_4^{(2)}(x)\right) &= I_2^{(1)} \otimes \Big(-2\ln(1+x)\Big) \\ &+ I_2^{(1)}(x) \otimes \Big(-2\ln(1+x) + 2\ln(x)\Big) \\ &+ I_4^{(1)}(x) \otimes \Big(\ln(1+x) - \ln(x)\Big)\end{aligned}$$

# The Duhr-Gangl-Rhodes Algorithm

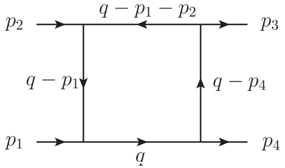
$$\Delta_{1,1}\left(I_4^{(2)}(x)\right) = -2a_1 \otimes \ln(1+x) + 2b_1 \otimes \ln(x) - 2b_1 \otimes \ln(1+x)$$
$$-c_1 \otimes \ln(x) + c_1 \otimes \ln(1+x) - 4(b_0 \ln(x)) \otimes \ln(x) + 4(b_0 \ln(x)) \otimes \ln(1+x)$$
$$+(c_0 \ln(x)) \otimes \ln(x) - (c_0 \ln(x)) \otimes \ln(1+x) + 2(a_0 \ln(1+x)) \otimes \ln(x)$$
$$-2(a_0 \ln(1+x)) \otimes \ln(1+x) + 2(b_0 \ln(1+x)) \otimes \ln(x) - 2(b_0 \ln(1+x)) \otimes \ln(1+x)$$
$$-(c_0 \ln(1+x)) \otimes \ln(x) + (c_0 \ln(1+x)) \otimes \ln(1+x)$$

The above expression can be used directly since the DGR algorithm suggests that $\{-x, 1/(1+x), x/(1+x)\}$ are the only allowed function arguments if one wishes to keep all basis functions manifestly real in the Euclidean region. At weight two, we actually have:

$$\left\{\ln^2(x), \ln(x)\ln(1+x), \ln^2(1+x), \mathrm{Li}_2(-x)\right\}$$

# Spurious Branch Cut Singularities Fix Constants

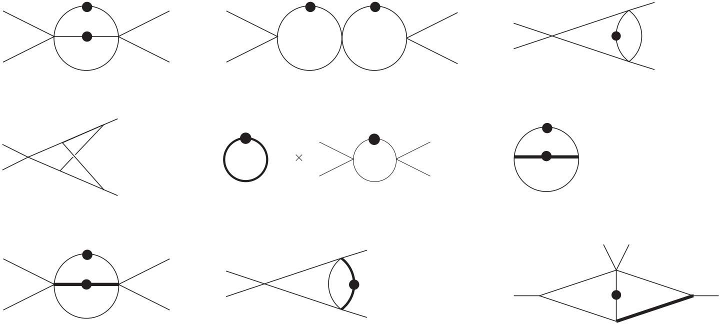
Our box integral:




has cuts in the $s$ and $t$ channels but its differential contains the letter $1 + x$. This implies the existence of a regularity condition at $x = -1$:

$$\frac{\mathrm{d}I_4^{(1)}(x)}{\mathrm{d}x} \to \frac{c_0 - 4}{1 + x} + \cdots \quad \text{as } x \to 1 \qquad \Longrightarrow \qquad c_0 = 4$$

# Integrals Not Fixed By Any Regularity Condition



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ALL integrals on the previous slide can be evaluated in closed form!

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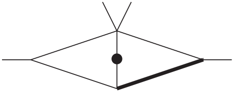 =

$$\frac{\Gamma(1+2\epsilon)\Gamma(2+\epsilon)\Gamma^2(-\epsilon)e^{2\gamma_E\epsilon}}{16m^4\Gamma(2-\epsilon)}\left(\frac{-s}{m^2}\right)^{2\epsilon} {}_3F_2\left(1,1,2+\epsilon;2,2-\epsilon;\frac{-t}{m^2}\right)$$
$$+\frac{\Gamma(1-\epsilon)\Gamma(2\epsilon)\Gamma^2(-\epsilon)e^{2\gamma_E\epsilon}}{16m^2(-t)\Gamma(1-3\epsilon)}\left(\frac{-s}{-t}\right)^{2\epsilon} {}_3F_2\left(1,-2\epsilon,1-\epsilon;1-2\epsilon,1-3\epsilon;\frac{-t}{m^2}\right)$$

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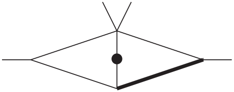 =

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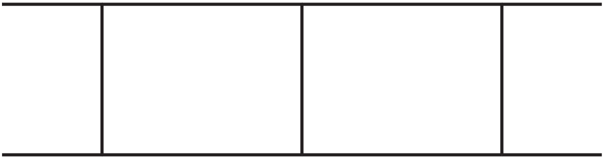
Alternatively, one can also use a convenient direct integration method to compute the input integrals order-by-order in $\epsilon$ (see Andreas von Manteuffel's talk Friday based on our work with Erik Panzer).

# Fixing Constants At The Multi-Loop Level

Even well-known approaches like exploiting unitarity are subtle at the multi-loop level. Case in point:



Anastasiou *et. al.*, Nucl. Phys. **B580**, 577, 2000

To get a feeling, we recommend a paper by Gehrmann *et. al.*, JHEP 1406, 032, 2014, as well as two papers by Henn *et. al.*, JHEP 1307, 128, 2013 and JHEP 1403, 088, 2014. Finally, in certain situations, we found that knowing how integrals scale in asymptotic limits could be useful (see Jantzen *et. al.*, Eur. Phys. J. **C72**, 2139, 2012).

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- Numerical checks in physical kinematics using `FIESTA 3` (A. V. Smirnov, Comput. Phys. Commun. **185**, 2090, 2014) and `SecDec 3` (S. Borowka *et. al.*, arXiv:1502.06595), powered respectively by the `VEGAS` and `DIVONNE` routines provided by the `CUBA` library (T. Hahn, Comput. Phys. Commun. **168**, 78, 2005).

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- Compute the two-mass master integrals in our new framework.
- Put the virtuals together and then collect the remaining ingredients needed to compute observables.
- Treat similar processes. As Stefan Dittmaier pointed out in his talk, Drell-Yan-like $W$ production is very important as well.