LOOP-TREE DUALITY AND QUANTUM FIELD THEORY IN 4D

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Content

- Introduction (motivation and notation)
- Loop-tree duality (LTD) (quick review)
- IR regularization
  - Threshold and IR singularities
  - Finite real+virtual integration
- UV renormalization
- Conclusions

Based in:

Catani et al., JHEP 09 (2008) 065
Bierenbaum et al., JHEP 1010 (2010) 073; JHEP 03 (2013) 025
Buchta et al., JHEP 11 (2014) 014
Rodrigo et al., in preparation (to be published soon…)
Collider physics are dominated by QCD effects

Exact solutions for scattering amplitudes in gauge theories are unknown (i.e. no exact solutions for QCD...) Use perturbation theory!

We need to reach higher orders in pQCD (increase theoretical accuracy)

Deal with ill-defined expressions in intermediate steps DREG!!!

- Proposed by {Giambiagi&Bollini, t’Hooft&Veltman, Cicuta&Montaldi, Ashmore,...}, it becomes a standard in HEP since it preserves gauge invariance
- Change the number of space-time dimensions from 4 to $d = 4 - 2\varepsilon$
- Singularities regularized as poles in $\varepsilon$ (meromorphic functions)

Final physical predictions must be FINITE Singularities MUST cancel!
Introduction

- Two kinds of physical singularities:
  - **Ultraviolet singularities** coming from the high-energy region of the loop momentum, due to the pointlike structure of the theory
    
    Add proper counterterms obtained from RENORMALIZATION procedure. These counterterms have EXPLICIT $\varepsilon$-poles and are proportional to lower-order amplitudes.
  
  - **Infrared singularities** associated with degenerate configurations: extra-particle radiation in soft (i.e. low energy) or collinear (i.e. parallel) configurations
    
    Kinoshita-Lee-Nauenberg theorem states that adding real-emission processes and computing IR-safe observables guarantees the cancellation of all the IR poles present in renormalized virtual amplitudes and INTEGRATED real-radiation contributions. ADD suitable SUBTRACTION TERMS to virtual and real corrections independently.

- **Threshold singularities** are integrable; countour deformation techniques can be applied to obtain numerically stable integrals.
Loop-tree duality (LTD)

Introduction and motivation

- KLN theorem suggests that virtual and real contributions have the same IR divergent structure (because they cancel in IR-safe observables).
- Cut contributions are similar to tree-level scattering amplitudes, if all the loops are cut. At one-loop, 1-cuts are tree-level objects (higher-cuts are products of unconnected graphs).
- Objective: Combine real and virtual contributions at integrand level and perform the computation in four-dimensions (take $\varepsilon$ to 0 with DREG).
  - Write virtual contributions as real radiation phase-space integrals of «tree-level» objects $1$-cut = sum over «tree level» contributions
  - Loop measure is related with extra-radiation phase-space

Loop-Tree Duality

Catani et al, JHEP 09 (2008) 065
Loop-tree duality (LTD)

Feynman integrals and propagators

Generic one-loop Feynman integral

Momenta definition

\[
L_R^{(N)}(p_1, p_2, \cdots, p_N) = -i \int \frac{d^d \ell}{(2\pi)^d} \prod_{i=1}^{N} G_R(q_i).
\]

\[
q_i = \ell + \sum_{k=1}^{i} p_k
\]

+i0 prescription guarantees that positive (negative) frequencies are propagated forward (backward) in time

Feynman propagator

\[
G(q) \equiv \frac{1}{q^2 + i0}
\]

q_0(q_{\pm}) plane
**Loop-tree duality (LTD)**

**Cauchy Residue theorem**

\[
\oint_{\gamma} f(z) \, dz = 2\pi i \sum \text{Res}(f, a_k)
\]

«If \( f \) is a holomorphic function in \( U/\{a_i\} \), and \( \gamma \) a simple positively oriented curve, then the integral is given by the sum of the residues at each singular point \( a_i \)»

**Feynman propagator’s poles**

\[
[G(q)]^{-1} = 0 \implies q_0 = \pm \sqrt{q^2 - i0}
\]

**Location of Feynman’s integrand poles**
Loop-tree duality (LTD)

- **Idea:** «Sum over all possible 1-cuts» (but with a modified i0 prescription...)
  - Apply Cauchy residue theorem to the Feynman integral:
    \[
    L^{(N)}(p_1, p_2, \ldots, p_N) = \int_q \int dq_0 \prod_{i=1}^{N} G(q_i) = \int_q \int_{C_L} dq_0 \prod_{i=1}^{N} G(q_i) = -2\pi i \int_q \sum \text{Res}_{\{\text{Im} q_0 < 0\}} \left[ \prod_{i=1}^{N} G(q_i) \right]
    \]
  - Select the residue of the poles with negative imaginary part:
    \[
    \text{Res}_{\{i-\text{th pole}\}} \left[ \prod_{j=1}^{N} G(q_j) \right] = \left[ \text{Res}_{\{i-\text{th pole}\}} G(q_i) \right] \left[ \prod_{j=1}^{N} G(q_j) \right]_{\{i-\text{th pole}\}}
    \]
    \[
    \left[ \text{Res}_{\{i-\text{th pole}\}} \frac{1}{q_i^2 + i0} \right] = \int dq_0 \delta_+(q_i^2) \left[ \prod_{j \neq i} G(q_j) \right]_{\{i-\text{th pole}\}} = \prod_{j \neq i} \frac{1}{q_j^2 - i0 \eta(q_j - q_i)}
    \]

Set internal propagators on-shell

Introduction of «dual propagators» (\(h\) prescription, a future- or light-like vector)

Catani et al, JHEP 09 (2008) 065
Loop-tree duality (LTD)

- It is crucial to keep track of the $i0$ prescription! Duality relation involves the presence of dual propagators:

$$L^{(N)}(p_1, p_2, \ldots, p_N) = -\int \sum_{i=1}^{N} \tilde{\delta}(q_i) \prod_{j=1 \atop j \neq i}^{N} \frac{1}{q_j^2 - i0 \eta(q_j - q_i)}$$

- The prescription involves a future- or light-like vector (arbitrary) and could depend on the loop momenta (at 1-loop is always independent of $q$). It is related with the finite value of $I_0$ in intermediate steps...

- Implement a shift in each term of the sum to have the same measure: the loop integral becomes a phase-space integral!

$$\tilde{L}^{(N)}(p_1, p_2, \ldots, p_N) = I^{(N-1)}(p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_{N-1}) + \text{cyclic perms.}$$

$$= \sum_{i=1}^{N} I^{(N-1)}(p_i, p_i + p_{i+1}, \ldots, p_i + p_{i+1} + \cdots + p_{i+N-2})$$

$$I^{(n)}(k_1, k_2, \ldots, k_n) = \int_{q} \tilde{\delta}(q) \quad I^{(n)}(q; k_1, k_2, \ldots, k_n) = \int_{q} \tilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2qk_j + k_j^2 - i0 \eta k_j}$$

Basic dual phase-space integral

Catani et al, JHEP 09 (2008) 065
Loop-tree duality (LTD)

Summary for one-loop processes

Loop (Feynman integral)

\[ L^{(N)}(p_1, p_2, \ldots, p_N) = - \tilde{L}^{(N)}(p_1, p_2, \ldots, p_N) \]

Dual integral

\[ \tilde{L}^{(N)}(p_1, p_2, \ldots, p_N) = \int_q \sum_{i=1}^{N} \delta(q_i) \prod_{j=1 \atop j \neq i}^{N} \frac{1}{q_j^2 - i0 \eta(q_j - q_i)} \]

Sum of phase-space integrals!

Catani et al, JHEP 09 (2008) 065
Feynman integrands develop singularities when propagators go on-shell. LTD allows to understand it as soft/collinear divergences of real radiation.

- Forward (backward) on-shell hyperboloids associated with positive (negative) energy solutions.

\[
G_F(q_i)^{-1} = q_i^2 - m_i^2 + i0 = 0
\]

\[
q_i^{(\pm)} = \pm \sqrt{q_i^2 + m_i^2 \mp i0}
\]

- LTD equivalent to integrate along the forward on-shell hyperboloids.
- Degenerate to light-cones for massless propagators.
- Dual integrands become singular at intersections (two or more on-shell propagators)
LTD at one-loop

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IR regularization in LTD

- Reference example: Massless triangle in the time-like region

\[
L^{(1)}(p_1, p_2, -p_3) = \int \prod_{i=1}^{3} \frac{1}{q_i} = c_T \frac{\mu^{2\epsilon}}{\epsilon^2} (-s_{12} - i0)^{-1-\epsilon} = \sum_{1}^{3} I_i
\]

- \( I_1 = -\int \frac{\tilde{\delta}(q_1)}{(2q_1 \cdot p_2 - i0) (-2q_1 \cdot p_1 + i0)} \)
- \( I_2 = -\int \frac{\tilde{\delta}(q_2)}{(-2q_2 \cdot p_2 + i0) (-2q_2 \cdot p_{12} + s_{12} + i0)} \)
- \( I_3 = -\int \frac{\tilde{\delta}(q_3)}{(2q_3 \cdot p_1 - i0) (2q_3 \cdot p_{12} + s_{12} - i0)} \)

- This integral is UV-finite
- But, there are IR-singularities, associated to soft and collinear regions
- **OBJECTIVE:** Define a IR-regularized triangle integral by adding real corrections in 4D (i.e. no epsilon should appear at integrand level...)

IR regularization in LTD

- In order to analyze IR divergences and their origins, we parametrize the momenta involved in the dual representation.

\[
\begin{align*}
\hat{p}_1^\mu &= \frac{\sqrt{s_{12}}}{2} (1, 0, 0, 1) \\
\hat{p}_2^\mu &= \frac{\sqrt{s_{12}}}{2} (1, 0, 0, -1) \\
q_i^\mu &= \xi_{i,0} \frac{\sqrt{s_{12}}}{2} \left( 1, \sqrt{1 - y^2} \tilde{e}_T^i, y \right)
\end{align*}
\]

\[y \in [-1, 1], \quad \xi_{i,0} \in [0, \infty)\]

- Then, we rewrite dual integrals by using the new variables:

\[
\begin{align*}
I_1 &= \frac{1}{4s_{12}} \int d[\xi_{1,0}] d[v_1] \xi_{1,0}^{-1} (v_1 (1 - v_1))^{-1} \\
I_2 &= \frac{1}{4s_{12}} \int d[\xi_{2,0}] d[v_2] \frac{(1 - v_2)^{-1}}{1 - \xi_{2,0} + i0} \\
I_3 &= -\frac{1}{4s_{12}} \int d[\xi_{3,0}] d[v_3] \frac{v_3^{-1}}{1 + \xi_{3,0} - i0}
\end{align*}
\]

Measure in DREG (epsilon dependence comes from this...)

IR regularization in LTD

- Ananlize the integration region. Application of LTD converts loop-integrals into PS: integrate in forward light-cones.

- Only forward-backward interference originate threshold or IR poles.
- Forward-forward cancel among dual contributions
- Threshold and IR singularities associated with finite regions (i.e. contained in a compact region)
- No threshold or IR singularity at large loop momentum

- This structure suggests how to perform real-virtual combination! Also, how to overcome threshold singularities (integrable, but numerically instable…)

IR regularization in LTD

- Analize the integration region. Application of LTD converts loop-integrals into PS: integrate in forward light-cones.

  - Only **forward-backward** interference originate **threshold or IR poles**.
  - **Forward-forward** cancel among dual contributions
  - Threshold and IR singularities associated with finite regions (i.e. contained in a compact region)
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From the previous plots, we define three contributions:

**IR-divergent contributions** \((\xi_0 < 1 + w)\)
- Originated in a **finite region** of the loop three-momentum
- All the IR singularities of the original loop integral

\[
I^{\text{IR}} = I_1^{(s)} + I_1^{(c)} + I_2^{(c)} = c_T \frac{1}{s_{12}} \left( \frac{-s_{12} - i0}{\mu^2} \right)^{-\epsilon} \times \left[ \frac{1}{\epsilon^2} + \left( \ln^2(2) \ln(w) - \frac{\pi^2}{3} - 2 \text{Li}_2 \left( -\frac{1}{w} \right) + \pi \ln(2) \right) \right] + O(\epsilon)
\]

**Forward integrals** \((v < 1/2, \xi_0 > 1)\)
- Free of IR/UV poles
- Integrable in 4-dimensions!

\[
I^{(f)} = \sum_{i=1}^{3} I_i^{(f)} = c_T \frac{1}{s_{12}} \left[ \frac{\pi^2}{3} - \pi \log(2) \right] + O(\epsilon)
\]

**Backward integrals** \((v > 1/2, \xi_0 > 1 + w)\)
- Free of IR/UV poles
- Integrable in 4-dimensions!

\[
I^{(b)} = c_T \frac{1}{s_{12}} \left[ 2 \text{Li}_2 \left( -\frac{1}{w} \right) - \ln^2(2) \ln(w) \right] + O(\epsilon)
\]
Let’s stop and make some remarks about the structure of these expressions:

- Introduction of an **arbitrary cut** \( w \) to **include threshold regions**.
- Forward and backward integrals can be performed in 4D because the sum does not contain poles.
- Presence of extra Log’s in (F) and (B) integrals. They are originated from the expansion of the measure in DREG, i.e.

\[
\xi^{-1-2\epsilon} = \frac{-Q_s^{2\epsilon}}{2\epsilon} \delta(\xi) + \left( \frac{1}{\xi} \right)_C - 2\epsilon \left( \frac{\ln(\xi)}{\xi} \right)_C + \mathcal{O}(\epsilon^2)
\]

for both \( v \) and \( \bar{\xi} \) (keep finite terms only). **It is possible to avoid them!**

- **IR-poles isolated in \( I^{IR} \)**. **IR divergences originated in compact region of the three-loop momentum!!!**

\[
L^{(1)}(p_1, p_2, -p_3) = I^{IR} + I^{(b)} + I^{(f)}
\]

Explicit poles
still present...

Can be done in 4D!

---

Now, we must add real contributions. Suppose one-loop scalar scattering amplitude given by the triangle

\[ |\mathcal{M}^{(0)}(p_1, p_2; p_3)\rangle = ig \]
\[ |\mathcal{M}^{(1)}(p_1, p_2; p_3)\rangle = -ig^3 L^1(p_1, p_2, -p_3) \]  \( \Rightarrow \) \( \text{Re} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \rangle \)

1->2 one-loop process \( \rightarrow \) 1->3 with unresolved extra-parton

Add scalar tree-level contributions with one extra-particle; consider interference terms:

\[ |\mathcal{M}_{ir}^{(0)}(p'_1, p'_2, p'_r; p_3)\rangle = -ig^2 / s_{ir} \Rightarrow \text{Re} \langle \mathcal{M}_{ir}^{(0)} | \mathcal{M}_{jr}^{(0)} \rangle = \frac{g^4}{s_{ir} s_{jr}} \]

Generate 1->3 kinematics starting from 1->2 configuration plus the loop three-momentum \( \vec{l} \) !!!

IR regularization in LTD

- **Mapping of momenta:** generate 1->3 real emission kinematics (3 external on-shell momenta) starting from the variables available in the dual description of 1->2 virtual contributions (2 external on-shell momenta and 1 free three-momentum)

\[
p_1^{\mu} = q_1^{\mu} \quad p_2^{\mu} = -q_3^{\mu} + \alpha_1 p_2^{\mu} = p_1^{\mu} - q_1^{\mu} + \alpha_1 p_2^{\mu} \quad p_2^{\mu} = (1 - \alpha_1) p_2^{\mu}
\]
\[
\alpha_1 = \frac{q_3^2}{2q_3 \cdot p_2} \quad q_1 = \ell + p_1
\]

- Mapping optimized for \( y'_{1r} < y'_{2r} \); analogous expression in the complement
- Express interference terms using this map

\[
\tilde{\sigma}_{i,R} = \sigma_0^{-1} 2\text{Re} \int d\Phi_1 \to 3 \langle M_{2r}^{(0)} | M_{1r}^{(0)} \rangle \theta(y'_{jr} - y'_{ir})
\]
\[
\tilde{\sigma}_{i,V} = \sigma_0^{-1} 2\text{Re} \int d\Phi_1 \to 2 \langle M^{(0)} | M_{i}^{(1)} \rangle \theta(y'_{jr} - y'_{ir})
\]

Real and virtual contributions are described using the same integration variables!

- Only required for \( I_1 \) and \( I_2 \) (\( I_3 \) singularities cancel among dual terms)

\[
\tilde{\sigma}_1 = \tilde{\sigma}_{1,V} + \tilde{\sigma}_{1,R} = \mathcal{O}(\epsilon)
\]
\[
\tilde{\sigma}_2 = \tilde{\sigma}_{2,V} + \tilde{\sigma}_{2,R} = -c_\Gamma \frac{g^2}{s_{12}} \frac{\pi^2}{6} + \mathcal{O}(\epsilon)
\]

Reference example: bubble with massless propagators

\[ L^{(1)}(p, -p) = \int\prod_{i=1}^{2} G_F(q_i) = \frac{c_T}{\epsilon (1 - 2\epsilon)} \left( \frac{-p^2 - i0}{\mu^2} \right)^{-\epsilon} = \sum_{i=1}^{2} I_i \]

\[ I_1 = -\int_{\ell} \tilde{\delta}(q_1) G_D(q_2; q_1) = -\int_{\ell} \frac{\tilde{\delta}(q_1)}{p^2 - 2q_1 \cdot p + i0} \]
\[ I_2 = -\int_{\ell} \tilde{\delta}(q_2) G_D(q_1; q_2) = -\int_{\ell} \frac{\tilde{\delta}(q_2)}{p^2 + 2q_2 \cdot p - i0} \]

- In this case, the integration regions of dual integrals are two energy-displaced forward light-cones. This integral contains UV poles only.

- **OBJECTIVE:** Define a **UV-regularized** triangle integral by adding unintegrated UV counter-terms, and find a purely 4-dimensional integral that represents this regularized bubble!

Divergences arise from the high-energy region (UV poles) and must be cancelled with a suitable renormalization counter-term. For the scalar case, we use

\[ I_{UV}^{cnt} = \int \frac{1}{\ell} \frac{1}{(q_{UV}^2 - \mu_{UV}^2 + i0)^2} \]


Dual representation (new: double poles in the loop energy)

\[ I_{UV}^{cnt} = \int \frac{\tilde{\delta}(q_{UV})}{\ell} \frac{1}{2 (q_{UV,0}^{(+)} + i0)^2} \]

Bierenbaum et al., JHEP 03 (2013) 025

Loop integration for loop energies larger than \( \mu_{UV} \)

UV renormalization in LTD

Cancellation of UV singularities

- Using the standard parametrization we define

\[ L^{(1)}(p, -p) - I_{UV}^{cnt} = c_T \left[ -\log \left( \frac{p^2}{\mu_{UV}^2} - i0 \right) + 2 \right] + \mathcal{O}(\epsilon) \]

- Since it is finite, we can express the regularized bubble in terms of 4-dimensional quantities (i.e. no epsilon required!!)

- **Physical interpretation of renormalization scales:** Separation between on-shell hyperboloids in UV-counterterm is \( 2/\mu_{UV} \). To avoid intersections with forward light-cones associated with \( I_1 \) and \( I_2 \), the renormalization scale has to be larger or of the order of the hard scale. So, the minimal choice that fulfills this agrees with the standard choice (i.e. \( 1/2 \) of the hard scale).

Conclusions

- Introduced new method based on the Loop-Tree Duality (LTD) that allows to treat virtual and real contributions in the same way: simultaneous implementation and no need of IR subtraction

- Physical interpretation of IR/UV singularities in loop integrals

- Presented proof of concept of LTD with reference examples

**Perspectives:**

- Apply the technique to compute full NLO physical observables
- Extend the procedure to higher orders: NNLO and beyond
Thanks!!!