High-Energy Limit of QED
beyond Sudakov Approximation

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Topics discussed

- Introduction
  - high energy limit, power corrections and renormalization group

- QED vector form factor at high energy to $O(m_e^2/s)$
  - Sudakov and non-Sudakov double logarithms
  - soft electron pair emission and eikonal charge nonconservation
  - two-loop result, all order resummation and asymptotic formula
  - generalizations: QCD, scattering amplitudes, etc.

Based on:

Sudakov limit

- **Sudakov limit**
  - on-shell
  - exclusive
  - high energy
  - fixed angle

**Sudakov logarithms (leading power in \(1/Q^2\))**

- each external line gets \(e^{-\frac{\alpha}{4\pi} \frac{C_R}{2} \ln^2(Q^2)}\)

  Sudakov (1956); Frenkel, Taylor (1976)

- subleading logs exponentiate as well

  Mueller (1979); Collins (1980); Sen (1981); Sterman (1987); ...

**What about power suppressed logs?**
High energy limit beyond Sudakov approximation

- Logarithmically enhanced power corrections
  - can be phenomenologically important at moderate energies
  - can become asymptotically dominant
  - intriguing from QFT point of view

- Power corrections and renormalization group
  - OPE, large mass expansion $\Rightarrow$ local composite operators
  - threshold, nonrelativistic limit $\Rightarrow$ spatially nonlocal potentials
  - ultrarelativistic limit $\Rightarrow$ ?
QED form factor

\[ F = \bar{\psi}(p_1) \left( \gamma_\mu F_1 + \frac{i \sigma_{\mu\nu} q^\nu}{2 m_e} F_2 \right) \psi(p_2) \]

kinematics \[ p_1^2 = p_2^2 = m_e^2, \quad Q^2 = -(p_2 - p_1)^2 \gg m_e^2 \gg \lambda^2 \]

High-energy expansion

\[ F_1 = e^{-\frac{\alpha}{2\pi} \ln(\lambda^2 / m_e^2) \ln \rho} \sum_{n=0}^{\infty} \rho^n F_1^{(n)} \quad (\rho = m_e^2 / Q^2) \]

Sudakov double logarithmic result

\[ F_1^{(0)} = e^{-x} \quad (x = \frac{\alpha}{4\pi} \ln^2 \rho) \]
Dirac form factor to $\mathcal{O}(\rho)$

- **Main idea**
  - expansion by regions $\Rightarrow$ homogeneous integrals
  - leading singularity of a given region $\Rightarrow$ double log

- **Sudakov logs $\Rightarrow$ photon momentum** $l \to 0$
  - soft photons
  - eikonal electrons

\[
D = \frac{-g_{\mu\nu}}{l^2 - \lambda^2}
\]

\[
S \approx -\frac{p_i + m_e}{2p_i l}
\]

- either $\mathcal{O}(\rho^2)$ or subleading log

- **Cusp anomalous dimension**
  - Korchemsky, Radyushkin (1987)

\[
\Gamma_{cusp} = -\frac{\alpha}{\pi} \ln \rho \left( 1 + \mathcal{O}(\rho^2) \right)
\]
Dirac form factor to $\mathcal{O}(\rho)$

- Non-Sudakov logs $\Rightarrow l' = p_i - l \rightarrow 0$
  
  eikonal photons \hspace{1cm} soft electrons

$$D = \frac{g_{\mu\nu}}{2p_i l' - m_e^2 + \lambda^2} \hspace{1cm} S \approx \frac{m_e}{l''^2 - m_e^2}$$

Gorshkov, Gribov, Lipatov, Frolov (1968)

- Does not contribute at one-loop

- Shows up in two-loop non-planar diagram

  - $l'_1 = p_1 - l_1 \rightarrow 0$, \hspace{0.5cm} $l'_2 = p_2 - l_2 \rightarrow 0$

  $\Rightarrow$ does not destroy eikonal structure
Soft pair emission

“Twist” the diagram

- soft electron pair exchange
- eikonal charge nonconservation
Evaluation of double logs

- **New variables**
  - **Sudakov parameters** \( l = u p_1 + v p_2 + l_\perp \)
  - **hypercube coordinates** \( \eta = \ln v / \ln \rho, \ \xi = \ln u / \ln \rho, \ 0 < \eta, \ \xi < 1 \)

- **Double log coefficient**
  - **volume under hypersurface**
    \[
    F_1^{(1)} \bigg|_{2\text{-loop}} = -4 x^2 \int_{0<\eta_i, \ \xi_i<1} K(\eta_1, \eta_2, \xi_1, \xi_2) d\eta_1 d\eta_2 d\xi_1 d\xi_2
    \]
  - **kernel**
    \[
    K(\eta_1, \eta_2, \xi_1, \xi_2) = \theta(1-\eta_1-\xi_1) \theta(1-\eta_2-\xi_2) \theta(\eta_2-\eta_1) \theta(\xi_1-\xi_2)
    \]
  - **result**
    \[
    F_1^{(1)} \bigg|_{2\text{-loop}} = -x^2 / 3
    \]

Mastrolia, Remiddi (2003)

Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi (2005)
Universality

**Quasi-universal character**
- leading power corrections are due to soft pair exchange
- the coefficient depends on scattering details
- not determined by external particle charge only

**Two-loop form factors**
- Dirac, Pauli
- scalar, axial
- agree with known results
Resummation

- **Dressing with Sudakov photons**

(a) ![Diagram](a)

(b) ![Diagram](b)

(c) ![Diagram](c)

- no coupling to scalar quarks
Resummation

- **Multiple Sudakov exchange**

\[
F_1^{(1)} = -4x^2 \int d\eta_1 d\eta_2 d\xi_1 d\xi_2 \phi^b(\eta_1, \xi_2) \phi^c(\eta_1, \xi_1) \phi^c(\xi_2, \eta_2) \\
\times \left[ \phi^a(\eta_2, \xi_1) K_1(\eta_1, \eta_2, \xi_1, \xi_2) + K_2(\eta_1, \eta_2, \xi_1, \xi_2) \right]
\]

- **form factors:**

\[
\phi^a(\eta, \xi) = \exp \left[ -x(1 - \eta - \xi)^2 \right], \\
\phi^b(\eta, \xi) = \exp \left[ -2x\eta\xi \right], \\
\phi^c(\eta, \xi) = \exp \left[ x\eta(\eta + 2\xi - 2) \right],
\]
Resummation

Multiple Sudakov exchange

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\[ \times \left[ \phi^a(\eta_2, \xi_1)K_1(\eta_1, \eta_2, \xi_1, \xi_2) + K_2(\eta_1, \eta_2, \xi_1, \xi_2) \right] \]

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Kernels:

\[ K_1(\eta_1, \eta_2, \xi_1, \xi_2) = \theta(1 - \eta_2 - \xi_1)\theta(1 - \eta_2 - \xi_2) \times \theta(\eta_2 - \eta_1)\theta(\xi_1 - \xi_2), \]

\[ K_2(\eta_1, \eta_2, \xi_1, \xi_2) = \theta(1 - \eta_1 - \xi_1)\theta(1 - \eta_2 - \xi_2) \times \theta(\eta_2 - \eta_1)\theta(\xi_1 + \eta_2 - 1), \]
Resummation

• **The series**

\[
F_1^{(1)} = -\frac{x^2}{3} \left( 1 + \sum_{n=1}^{\infty} c_n x^n \right)
\]

• **The coefficients**

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n! n^3 c_n )</td>
<td>7/30</td>
<td>68/105</td>
<td>59/350</td>
<td>992/945</td>
<td>32225/29106</td>
<td>208044/175175</td>
<td>3946313/3281850</td>
</tr>
</tbody>
</table>

• **The asymptotics**

\[
c_n = \frac{C}{n^3 n!} \left[ 1 + O \left( \frac{1}{n^{1/2}} \right) \right]
\]

\( C = 1.1994 \ldots \)
High energy limit

- Large-$x$ behaviour

\[ F_1^{(1)} \sim e^{x - \ln x} \left[ -\frac{C}{3} + O\left( \frac{1}{x^{1/2}} \right) \right] \]

- exponential enhancement

- Leading term vs Power corrections

\[ F_1^{(0)} \sim \rho^{-\frac{\alpha}{4\pi}} \ln \rho \]
\[ \rho F_1^{(1)} \sim \rho^{1 + \frac{\alpha}{4\pi} \ln \rho} \]

- power correction determine the asymptotic for \(|\ln \rho| > \frac{2\pi}{\alpha}\)

below Landau pole, weak coupling \(\alpha(Q^2) \approx 3\alpha\)
The first resummation beyond Sudakov
Summary

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  - soft pair exchange
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  - eikonal charge nonconservation
  - DL enhancement
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- To be done
  - amplitudes beyond two legs
  - QCD beyond two loops
  - subleading logs