Simplified differential equations approach for NNLO calculations

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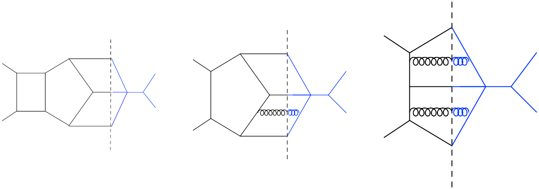
UCLA, June 19, 2015

What do we need for an NNLO calculation ?

$$p_1, p_2 \rightarrow p_3, \ldots, p_{m+2}$$



Perturbative QCD at NNLO

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$$\begin{aligned}
\sigma_{NNLO} \quad \to \quad & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) && VV \\
+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) && RV \\
+ \quad & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) && RR
\end{aligned}$$

$RV + RR \to$ Talks by M. Czakon, Currie, Abelof, Boughezal, Somogyi , ...

STRIPPER,...

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+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) & \quad RV \\
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Radcor > LoopFest

OPP at two loops

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$S_{m;n}$ stands for all subsets of m indices out of the n ones

$\rightarrow$ Talk by G.Ossola

The new "master" formula

$$\frac{N(q)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} = \sum_{i_0<i_1<i_2<i_3}^{m-1}\frac{d(i_0i_1i_2i_3)+\tilde{d}(q;i_0i_1i_2i_3)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}\bar{D}_{i2}}$$
$$+\sum_{i_0<i_1<i_2}^{m-1}\frac{c(i_0i_1i_2)+\tilde{c}(q;i_0i_1i_2)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}}$$
$$+\sum_{i_0<i_1}^{m-1}\frac{b(i_0i_1)+\tilde{b}(q;i_0i_1)}{\bar{D}_{i0}\bar{D}_{i1}}$$
$$+\sum_{i_0}^{m-1}\frac{a(i_0)+\tilde{a}(q;i_0)}{\bar{D}_{i0}}$$
$$+\ \text{rational terms}$$

`CutTools`, `Samurai`, `Ninja`, ... → Talk by G.Ossola

$$c_4\binom{m}{4}+c_3\binom{m}{3}+c_2\binom{m}{2}\sim C\,m^4$$

Reduction at the integrand level

Over the last few years very important activity to extend unitarity and integrand level reduction ideas beyond one loop

J. Gluza, K. Kajda and D. A. Kosower, "Towards a Basis for Planar Two-Loop Integrals," Phys. Rev. D **83** (2011) 045012 [arXiv:1009.0472 [hep-th]].

D. A. Kosower and K. J. Larsen, "Maximal Unitarity at Two Loops," Phys. Rev. D **85** (2012) 045017 [arXiv:1108.1180 [hep-th]].

P. Mastrolia and G. Ossola, "On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes," JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]].

S. Badger, H. Frellesvig and Y. Zhang, "Hepta-Cuts of Two-Loop Scattering Amplitudes," JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]].

Y. Zhang, "Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods," JHEP **1209** (2012) 042 [arXiv:1205.5707 [hep-ph]].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division," arXiv:1209.4319 [hep-ph].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes," arXiv:1307.5832 [hep-ph].

Multivariate Division and Groebner Basis

D. Cox, J. Little, D. O'Shea *Ideals, Varieties and Algorithms*

multivariate polynomial division

- Given any set of polynomials π_i, the ideal I, $f = \sum_i \pi_i h_i$, we can define a unique Groebner basis up to ordering $< g_1, \ldots, g_s >$

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$$f = h_1 g_1 + \ldots + h_n g_n + r$$

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Strategy:

• Start with a set of denominators, pick up a $4d$ parametrisation, define an ideal, $I =< D_1, \ldots, D_n >$, even D_i^2

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• Perform the division of an arbitrary polynomial N

$$N = h_1 g_1 + \ldots + h_n g_s + v$$

• Express back g_i in terms of D_i

$$N = \tilde{h}_1 D_1 + \ldots + \tilde{h}_n D_n + v$$

OPP AT TWO LOOPS

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$$\sum \frac{\Delta_{i_1 i_2 \dots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \dots D_{i_m}} \rightarrow \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

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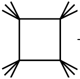
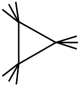
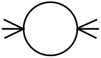
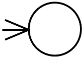
ISP-irreducible integrals → use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop` basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

D_0, C_0, B_0, A_0, scalar one-loop integrals: 't Hooft and Veltman
QCDLOOP Ellis & Zanderighi ; OneLOop A. van Hameren

S. Badger, talk in Amplitudes 2013

- Rational terms

$$l_1 \to l_1 + l_1^{(2\varepsilon)},\ \ l_2 \to l_2 + l_2^{(2\varepsilon)},\ \ l_{1,2} \cdot l_{1,2}^{(2\varepsilon)} = 0$$

$$\left(l_1^{(2\varepsilon)}\right)^2 = \mu_{11}, \quad \left(l_2^{(2\varepsilon)}\right)^2 = \mu_{22}, \quad l_1^{(2\varepsilon)} \cdot l_2^{(2\varepsilon)} = \mu_{12}$$

$$\left\{ l_1^{(4)}, l_2^{(4)} \right\} \to \left\{ l_1^{(4)}, l_2^{(4)}, \mu_{11}, \mu_{22}, \mu_{12} \right\}$$

Welcome: $I = \sqrt{I}$ prime ideal

OPP AT TWO LOOPS - RATIONAL TERMS

S. Badger, talk in Amplitudes 2013

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Welcome: $I = \sqrt{I}$ prime ideal

- R_2 terms: hopefully *à la* 1 loop

Rational Terms

Numerically treat $D = 4 - 2\epsilon$, means $4 \oplus 1$

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod^{m-1} \bar{D}_i + \tilde{P}(q) \prod^{m-1} \bar{D}_i
\end{aligned}$$

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\end{aligned}
$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

Rational Terms

Numerically treat $D = 4 - 2\epsilon$, means $4 \oplus 1$

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&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$m_i^2 \rightarrow m_i^2 - \tilde{q}^2$$

Rational Terms

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$
\begin{aligned}
\mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\
&- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) .
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

Rational Terms - R_2

A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2$$

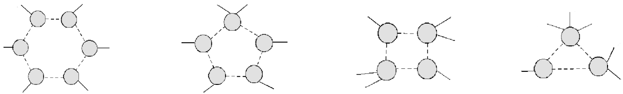
$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

New vertices

The one-loop calculation in a nutshell

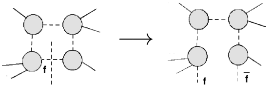
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, HELAC evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots$ with the values of the loop momentum q provided by CutTools

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

→ MadGraph, RECOLA, OpenLoops, Talks by S.Uccirati, V.Hirschi, ...

Towards a full NNLO solution

- From Feynman Diagrams to recursive equations: taming the $n!$.
  Dyson-Schwinger Recursive Equations

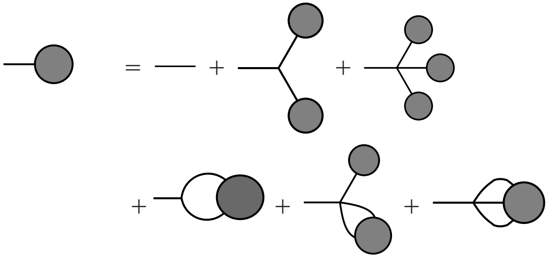
Towards a full NNLO solution

- From Feynman Diagrams to recursive equations: taming the $n!$. Dyson-Schwinger Recursive Equations

A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

S. Weinzierl

Integrand reduction at two loops

- Amplitude level (explicit color, helicity)
- Using 't Hooft-Veltman scheme, $d = 4$ ME
- Fully automated

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```
MG5_aMC> add process p p > e+ ve j j [QCD] @ 0
MG5_aMC> add process p p > e+ ve j j [QCD] @ 1
MG5_aMC> add process p p > e+ ve j j [QCD] @ 2
```

Master Integrals: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1,\ldots,a_N] = \int d^dk d^dl \frac{1}{D_1^{a_1}\ldots D_N^{a_N}}$$
  $$\int d^dk d^dl \;\; \frac{\partial}{\partial\{k^\mu, l^\mu\}}\left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1}\ldots D_N^{a_N}}\right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations

Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

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- $$F[a_1, \ldots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \ldots D_N^{a_N}}$$
  $$\int d^d k d^d l \quad \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \ldots D_N^{a_N}} \right) = 0$$
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- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k, l\} + p_i)^2 - M_i^2$
- $$F[a_1, \dots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \dots D_N^{a_N}}$$ $$\int d^d k d^d l \;\; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations

Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

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Master Integrals: The current approach

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Differential Equations Approach

- Iterated Integrals — K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

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C. Bogner and F. Brown

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with the special cases, $G(x)=1$ and

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Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
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- Boundary conditions: expansion by regions or regularity conditions.

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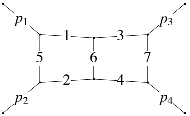
J. M. Henn, K. Melnikov and V. A. Smirnov, arXiv:1402.7078 [hep-ph].

T. Gehrmann, A. von Manteuffel and L. Tancredi, arXiv:1503.04812 [hep-ph].



$$S=(q_1+q_2)^2=(q_3+q_4)^2,\quad T=(q_1-q_3)^2=(q_2-q_4)^2,\quad U=(q_1-q_4)^2=(q_2-q_3)^2;$$

$$\frac{S}{M_3^2}=(1+x)(1+xy),\quad \frac{T}{M_3^2}=-xz,\quad \frac{M_4^2}{M_3^2}=x^2y.$$

$$d\,\vec{f}(x,y,z;\epsilon)=\epsilon\, d\,\tilde{A}(x,y,z)\,\vec{f}(x,y,z;\epsilon)$$

$$\tilde{A}=\sum_{i=1}^{15}\tilde{A}_{\alpha_i}\,\log(\alpha_i)$$

$$\alpha=\{x,y,z,1+x,1-y,1-z,1+xy,z-y,1+y(1+x)-z,xy+z,$$
$$1+x(1+y-z),1+xz,1+y-z,z+x(z-y)+xyz,z-y+yz+xyz\}.$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, arXiv:1401.6057 [hep-ph].

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k + p_1 + p_2)^2 \ldots (k + p_1 + p_2 + \ldots + p_n)^2}$$

- Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

- and using IBPI we obtain, for instance for the one-loop 3 off-shell legs

$$\begin{aligned} m_1 x G_{121} + \frac{1}{x} G_{021} &= \left(\frac{1}{x-1} + \frac{1}{x - m_3/m_1}\right)\left(\frac{d-4}{2}\right) G_{111} \\ &+ \frac{d-3}{m_1 - m_3}\left(\frac{1}{x-1} - \frac{1}{x - m_3/m_1}\right)\left(\frac{G_{101} - G_{110}}{x}\right) \end{aligned}$$

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The Simplified Differential Equations Approach

- The integrating factor M is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3+m_1x)^{\frac{4-d}{2}}$$

- and the DE takes the form, $d = 4 - 2\varepsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3+m_1x)^{-1+\varepsilon}\left(\left(-m_1x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

  - Integrating factors $\epsilon = 0$ do not have branch points
  - DE can be straightforwardly integrated order by order $\rightarrow$ GPs.

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The Simplified Differential Equations Approach

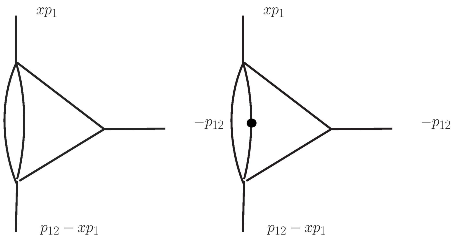
$$G_{111} = \frac{c_\Gamma}{(m_1 - m_3)x}\mathcal{I}$$

$$x_1 = (1-x)^{-\epsilon}$$

$$\begin{aligned}
\mathcal{I} = {} & \frac{-(-m_1)^{-\varepsilon} + (-m_3)^{-\varepsilon} + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) x_1}{\varepsilon^2} \\
& + \frac{\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) x_1 G\left(\frac{m_3}{m_1},1\right) - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},x\right)\right)}{\varepsilon} \\
& + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},1\right) G\left(\frac{m_3}{m_1},x\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right) + x_1 \Big(-2G(0,1,x)(-m_1)^{-\varepsilon} \\
& \quad + 2G\left(0,\frac{m_3}{m_1},x\right)(-m_1)^{-\varepsilon} + 2G\left(\frac{m_3}{m_1},1,x\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)(-m_1)^{-\varepsilon} - G\left(\frac{m_3}{m_1},x\right)\log(1-x)(-m_1)^{-\varepsilon} \\
& \quad - 2G\left(\frac{m_3}{m_1},x\right)\log(x)(-m_1)^{-\varepsilon} + 2\log(1-x)\log(x)(-m_1)^{-\varepsilon} - 2(-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1},1,x\right) - (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) \\
& \quad - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1},1\right)\left(G\left(\frac{m_3}{m_1},x\right) - \log(1-x)\right) + (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1},x\right)\log(1-x)\Big) \\
& + \varepsilon \Bigg(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\Big(G\left(\frac{m_3}{m_1},x\right) G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},1\right) G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) \\
& \quad + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\Big) + \frac{1}{2} x_1 \Big(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1},1\right)\left(\log^2(1-x) + 2G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right) \\
& \quad + G\left(\frac{m_3}{m_1},x\right)\Big(4\log^2(x) - 2\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) + 2\left((-m_3)^{-\varepsilon} - (-m_1)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1},1\right)\log(1-x)\Big) \\
& \quad + 2\left(G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\log(1-x)(-m_1)^{-\varepsilon} - 2\log(1-x)\log^2(x) - 4G(0,0,1,x)\right. \\
& + 4G\left(0,0,\frac{m_3}{m_1},x\right) - 2G(0,1,1,x) + 4G\left(0,\frac{m_3}{m_1},1,x\right) - 2G\left(0,\frac{m_3}{m_1},\frac{m_3}{m_1},x\right) + 2G\left(\frac{m_3}{m_1},0,1,x\right) - 2G\left(\frac{m_3}{m_1},0,\frac{m_3}{m_1},x\right) \\
& - (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\log(1-x) + \log^2(1-x)\log(x) \\
& \left. + 4G(0,1,x)\log(x) - 4G\left(0,\frac{m_3}{m_1},x\right)\log(x) - 4G\left(\frac{m_3}{m_1},1,x\right)\log(x) + 2G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\log(x)\right)\Big)\Bigg)
\end{aligned}$$

The Simplified Differential Equations Approach

The two-loop 3-off-shell-legs triangle



The Simplified Differential Equations Approach

We are interested in $G_{0101011}$. The DE involves also the MI $G_{0201011}$, so we have a system of two coupled DE, as follows:

$$\frac{\partial}{\partial x} f(x) = \frac{A_3(2-3\varepsilon)(1-x)^{-2\varepsilon} x^{-1+\varepsilon} (m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon(2\varepsilon-1)} + \frac{m_1 \varepsilon (1-x)^{-2\varepsilon} (m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon - 1} g(x)$$

$$\frac{\partial}{\partial x} g(x) = \frac{A_3(3\varepsilon-2)(3\varepsilon-1)(-m_1)^{-2\varepsilon}(1-x)^{2\varepsilon-1} x^{-3\varepsilon} (m_1 x - m_3)^{2\varepsilon-1}}{2\varepsilon^2} + (2\varepsilon-1)(3\varepsilon-1)(1-x)^{2\varepsilon-1}(m_1 x - m_3)^{2\varepsilon-1} f(x)$$

where $f(x) \equiv M_{0101011} G_{0101011}$ and $g(x) \equiv M_{0201011} G_{0201011}$, $M_{0201011} = (1-x)^{2\varepsilon} x^{\varepsilon+1} (m_1 x - m_3)^{2\varepsilon}$ and $M_{0101011} = x^{\varepsilon}$

- Solve sequentially in ε expansion
- Reproduce limit $\varepsilon \to 0$

The Simplified Differential Equations Approach

The singularity at $x = 0$ is proportional to $x^{-1+\varepsilon}$ and can easily be integrated by the following decomposition

$$\int_0^x dt\ \ t^{-1+\varepsilon} F(t) = F(0) \int_0^x dt\ \ t^{-1+\varepsilon} + \int_0^x dt\ \ \frac{F(t)-F(0)}{t} t^{\varepsilon}$$
$$= F(0)\,\frac{x^{\varepsilon}}{\varepsilon} + \int_0^x dt\ \ \frac{F(t)-F(0)}{t}\left(1 + \varepsilon \log(t) + \frac{1}{2}\varepsilon^2 \log^2(t) + \ldots\right)$$

Reproduce correctly boundary term $x = 0$

The Simplified Differential Equations Approach

General setup

m: number of denominators

$$\partial_x G_{m+1} = H\left(\{s_{ij}\}, \epsilon; x\right) G_{m+1} + \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'} ,$$

$m_0 = 3$ in the case of two loops

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$$M \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'} =: \sum_i x^{-1+\beta_i \epsilon} \tilde{I}_{sin}^{(i)}(\{s_{ij}\}, \epsilon) + \tilde{I}_{reg}(\{s_{ij}\}, \epsilon; x).$$

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i\epsilon}}{\beta_i\epsilon}\tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx'\tilde{I}_{reg}(\{s_{ij}\},\epsilon;x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

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- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

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The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D$: $\partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $\sim C\, x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

more details → talk by C. Wever

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

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Two-loop, four-point, two off-shell legs

$$p(q_1)p'(q_2) \to V_1(-q_3)V_2(-q_4), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = M_4^2.$$

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$$q_1 = xp_1, \quad q_2 = xp_2, \quad q_3 = p_{123} - xp_{12}, \quad q_4 = -p_{123}, \quad p_i^2 = 0,$$
$$s_{12} := p_{12}^2, \qquad s_{23} := p_{23}^2, \qquad q := p_{123}^2,$$

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$$S = (q_1 + q_2)^2 \qquad T = (q_1 + q_3)^2$$

$$S = s_{12}x^2, \quad T = q - (s_{12} + s_{23})x, \quad M_3^2 = (1-x)(q - s_{12}x), \quad M_4^2 = q.$$

$$U = (q_1 + q_4)^2 : \; S + T + U = M_3^2 + M_4^2.$$

Two-loop, four-point, two off-shell legs

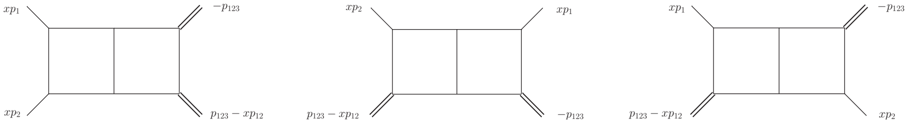

FIGURE : The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

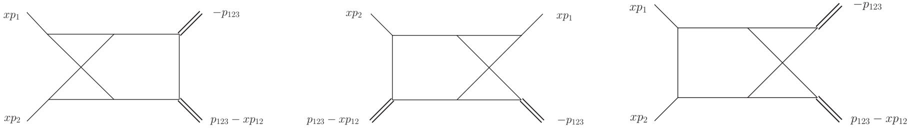

FIGURE : The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

Two-loop, four-point, two off-shell legs

Triangle rule:

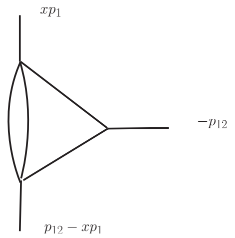

Figure : Required parametrization for off mass-shell triangles after possible pinching of internal line(s).

more details → talk by C. Wever

Two-loop, four-point, two off-shell legs

Planar topologies

$$G^{P_{12}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{P_{13}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
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$$G^{P_{23}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

Two-loop, four-point, two off-shell legs

Planar topologies

P_{12} : {010000011, 001010001, 001000011, 100000011, 101010010, 101010100, 101000110, 010010101, 101000011, 101000012, 100000111, 100000112, 001010011, 001010012, 010000111, 010010011, 101010110, 111000011, 101000111, 101010011, 011010011, 011010012, 110000111, 110000112, 010010111, 010010112, 111010011, 111000111, 111010111, 111m10111, 11101m111},

P_{13} : {000110001, 001000011, 001010001, 001101010, 001110010, 010000011, 010101010, 010110010, 001001011, 001010011, 001010012, 001011011, 001101001, 001101011, 001110001, 001110002, 001110011, 001111001, 001111011, 001211001, 010010011, 010110001, 010110011, 011010011, 011010021, 011110001, 011110011, 011111011, m11111011},

P_{23} : {001010001, 001010011, 010000011, 010000101, 010010011, 010010101, 010010111, 011000011, 011010001, 011010010, 011010011, 011010012, 011010100, 011010101, 011010111, 011020011, 012010011, 021010011, 100000011, 101000011, 101010010, 101010011, 101010100, 110000111, 111000011, 111010011, 111010111, 111m10111}.

Two-loop, four-point, two off-shell legs

Non-planar topologies

$$G^{N12}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
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$$G^{N13}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-xp_{12})^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_1)^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{N34}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_{12}-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}.$$

Non-planar topologies

N_{12} : {100001010, 000110010, 000110001, 000101010, 000101001, 101010010, 100110010, 100101020, 100101010, 100101001, 001110010, 001110002, 001110001, 001101001, 101110020, 101110010, 101101002, 101101001, 100111020, 100111010, 100102011, 100101011, 001120011, 001111002, 001111001, 001110011, 000111011, 101011011, 100111011, 1m0111011, 001111011, 0m1111011, 101111011, 1m1111011, 1m1111m11},

N_{13} : {010000110, 000110010, 001000101, 001000110, 001010001, 010110100, 001110100, 001010102, 001110002, 000110110, 001010101, 001010110, 001100110, 001110001, 001110010, 010100110, 010110101, 002010111, 001120011, 001210110, 011010102, 001110120, 001010111, 001110210, 001110011, 001110101, 001110110, 002110110, 011000111, 011010101, 011100110, 011110001, 011110110, m11010111, 010110111, m01110111, 0m1110111, 00111m111, 001110111, 011010111, 011110101, 011110111, m11110111},

N_{34} : {001001010, 001010010, 010010010, 100000110, 100010010, 000010111, 010010110, 001010102, 001010101, 010010101, 001020011, 010000111, 001010011, 010010011, 101010020, 101010010, 101010100, 101000011, 110010120, 110010110, 010010112, 010010121, 010010111, 010020111, 020010111, 011010102, 001010111, 011010101, 110000211, 011020011, 110000111, 011010011, 111000101, 111010010, 101010101, 101010011, 111010110, 111010101, 101010111, 11m010111, 110m10111, 11001m111, 110010111, m11010111, 011m10111, 01101m111, 011010111, 111000111, 111010011, 111010111, 111m10111}.

Two-loop, four-point, two off-shell legs

GP-indices

$$I(P_{12}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}}{q}, \frac{q}{q - s_{23}}, 1 - \frac{s_{23}}{q}, 1 + \frac{s_{23}}{s_{12}}, \frac{s_{12}}{s_{12} + s_{23}}\right\},$$
$$I(P_{13}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12} + s_{23}}{s_{12}}, \frac{q}{q - s_{23}}, \xi_-, \xi_+, \frac{q(q - s_{23})}{q^2 - (q + s_{12})s_{23}}\right\},$$
$$I(P_{23}) = \left\{0, 1, \frac{q}{s_{12}}, 1 + \frac{s_{23}}{s_{12}}, \frac{q}{q - s_{23}}, \frac{q}{s_{12} + s_{23}}, \frac{q - s_{23}}{s_{12}}\right\},$$
$$\xi_\pm = \frac{q s_{12} \pm \sqrt{q s_{12} s_{23}(-q + s_{12} + s_{23})}}{q s_{12} - s_{12} s_{23}}.$$

$$I(N_{12}) = I(P_{23}),$$
$$I(N_{34}) = I(P_{12}) \cup I(P_{23}) \cup \left\{\frac{s_{12}}{q - s_{23}}, \frac{s_{12} + s_{23}}{q}, \frac{q^2 - q s_{23} - s_{12} s_{23}}{s_{12}(q - s_{23})}, \frac{s_{12}^2 + q s_{23} + s_{12} s_{23}}{s_{12}(s_{12} + s_{23})}\right\},$$
$$I(N_{13}) = I(P_{23}) \cup \left\{\xi_-, \xi_+, 1 + \frac{q}{s_{12}} + \frac{q}{-q + s_{23}}\right\}.$$

Two-loop, four-point, two off-shell legs

Example

$$G^{P_{13}}_{011111011}(x,s,\epsilon) = \frac{A_3(\epsilon)}{x^2 s_{12}(-q+x(q-s_{23}))^2}\left\{\frac{-1}{2\epsilon^4} + \frac{1}{\epsilon^3}\left(-GP\left(\frac{q}{s_{12}};x\right) + 2\,GP\left(\frac{q}{q-s_{23}};x\right)\right.\right.$$

$$\left. +2\ GP(0;x) - GP(1;x) + \log\left(-s_{12}\right) + \frac{9}{4}\right) + \frac{1}{4\epsilon^2}\left(18\ GP\left(\frac{q}{s_{12}};x\right) - 36\ GP\left(\frac{q}{q-s_{23}};x\right)\right.$$

$$\left. -8\ GP\left(0,\frac{q}{s_{12}};x\right) + 16\ GP\left(0,\frac{q}{q-s_{23}};x\right) + 8\ GP\left(\frac{s_{23}}{s_{12}}+1,\frac{q}{q-s_{23}};x\right) + \cdots\right)$$

$$+\frac{1}{\epsilon}\left(9\left(GP\left(0,\frac{q}{s_{12}};x\right) + GP(0,1;x)\right) - 4\left(GP\left(0,0,\frac{q}{s_{12}};x\right) + GP(0,0,1;x)\right) + \cdots\right)$$

$$\left. +6\left(GP\left(0,0,1,\xi_-;x\right) + GP\left(0,0,1,\xi_+;x\right)\right) - 2\,GP\left(0,0,\frac{q}{q-s_{23}},\frac{q\left(q-s_{23}\right)}{q^2-s_{23}\left(q+s_{12}\right)};x\right) + \cdots\right\}.$$

$$A_3(\epsilon) = -e^{2\gamma_E\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(3-3\epsilon)}.$$

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1409.6114 [hep-ph].

The Simplified Differential Equations Approach

Summary & Outlook

- One-loop up to 5-point at order ϵ: 6 scales, GP-weight 3(4) (look forward for pentaboxes)
- Two-loop triangles and 4-point MI
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NNLO in the future

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  - In a few years the new "wish list" should be completed
    $pp \to t\bar{t}$, $pp \to W^+W^-$, $pp \to W/Z + nj$, $pp \to H + nj$, ...
  - Virtual amplitudes: Reduction at the integrand level $\oplus$ IBP

    $\to$ Master Integrals
  - Virtual-Real & Real-Real

STRIPPER, M. Czakon, Phys. Lett. B **693** (2010) 259 [arXiv:1005.0274 [hep-ph]].

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  - In a few years the new "wish list" should be completed
    $pp \to t\bar{t}$, $pp \to W^+W^-$, $pp \to W/Z + nj$, $pp \to H + nj$, $\ldots$
  - Virtual amplitudes: Reduction at the integrand level $\oplus$ IBP

    $\to$ Master Integrals
  - Virtual-Real & Real-Real

STRIPPER, M. Czakon, Phys. Lett. B **693** (2010) 259 [arXiv:1005.0274 [hep-ph]].

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NNLO IN THE FUTURE

NNLO: the next frontier.
These are the voyages at Radcor-Loopfest conferences.
Its five-year mission:
to explore strange new subtraction schemes,
to seek out new methods to calculate Master Integrals
and new tools to perform Integrand Reduction,
to boldly go where no man has gone before.

Thanks !