Linear reducibility of Feynman integrals

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European Research Council
Established by the European Commission

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RADCOR/LoopFest 2015
June 19th, 2015
UCLA, Los Angeles

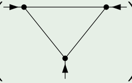
Feynman integrals: special functions and numbers

Some FI are expressible via multiple polylogarithms (MPL)

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \sum_{0<k_1<\cdots<k_d} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}$$

Example ($p_1^2 = 1$, $p_2^2 = z\bar{z}$, $p_3^2 = (1-z)(1-\bar{z})$)

$$\Phi\left(\right.$$



$$\left.\right) = \frac{4i\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{z-\bar{z}}$$

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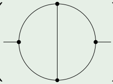
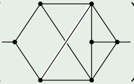
Example (massless propagators)

$\Phi\Big($  $\Big) = 6\zeta_3,\quad \Phi\Big($  $\Big) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

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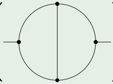
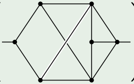
$\Phi\Bigg($  $\Bigg) = 6\zeta_3,\quad \Phi\Bigg($  $\Bigg) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

Questions

1. What comes beyond MPL? [Weinzierl's talk]
2. How to tell if a FI evaluates to MPL? What is the alphabet?
3. How to compute it explicitly?

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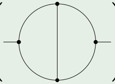
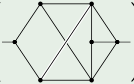
$\Phi\Bigg($  $\Bigg) = 6\zeta_3$, $\Phi\Bigg($  $\Bigg) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

Questions

1. What comes beyond MPL? [Weinzierl's talk]
2. How to tell if a FI evaluates to MPL? What is the alphabet?
3. How to compute it explicitly in an automated way?

Tools to study Feynman integrals

- integral representations
  - momentum space [IBP, unitarity]
  - position space [Gardi's talk, Drummond & Schnetz graphical functions]
  - twistor space [Trnka's talk]
  - parametric space [this talk]
  - Mellin-Barnes
- sum representations [Schneider's talk]
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Plan of the talk

1. predicting bounds on the singularities of an integral
2. nice (linearly reducible) integral representations can be integrated in terms of MPL
3. good parametrizations for FI via Laplace transform

Definite integral representations

- hypergeometric functions ${}_pF_q(\cdots;z)$

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t $$

- Appell's functions F_1, F_2, F_3, F_4

$$ F_3\left(\begin{matrix} \alpha,\alpha' \\ \beta,\beta' \end{matrix}\middle|\gamma\middle| x\ y\right) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} $$
$$ \times \int_0^1 \int_0^{1-v} u^{\beta-1} v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}\mathrm{d}u\mathrm{d}v $$

- Feynman integrals in Schwinger parameters
- Phase-space integrals
- ...

Schwinger parameters

With the *superficial degree of divergence* $\mathrm{sdd} = |E(G)| - D/2 \cdot \mathrm{loops}(G)$,

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_{(0,\infty)^E} \frac{1}{\psi^{D/2}} \left(\frac{\psi}{\varphi}\right)^{\mathrm{sdd}} \delta(1-\alpha_N) \prod_e \alpha_e^{a_e - 1} \mathrm{d}\alpha_e$$

Graph polynomials:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e$$

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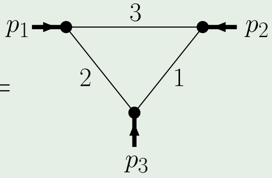
$G =$ 

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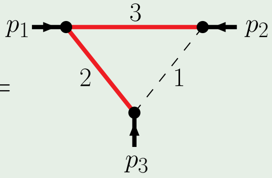
$G =$ 

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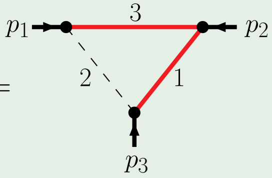
$G =$ 

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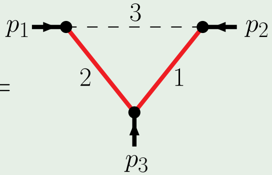
$G =$ 

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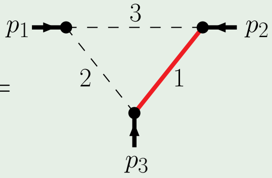

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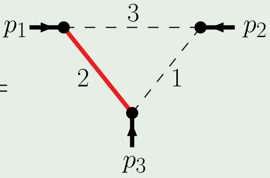
$G =$ 

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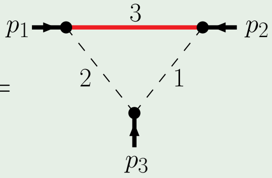

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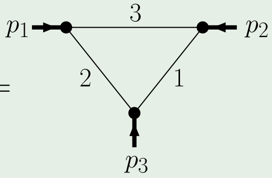
$G =$ 

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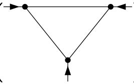
$\Phi\Bigg($  $\Bigg) = \displaystyle\int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$

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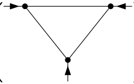
$\Phi\Bigg($  $\Bigg) = \displaystyle\int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$

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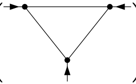
$$\Phi\left(\right.$$  $$\left.\right) = \int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$$

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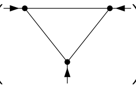
$\Phi\Bigg($  $\Bigg) = \displaystyle\int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$

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With the same logic, predict the possible singularities after $\int_0^\infty \mathrm{d}\alpha_2$:

$$S_{3,2} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-1\}$$

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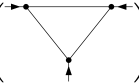
$\Phi\Big($  $\Big)$

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Polynomial reduction [F. Brown]

Definition

Let S denote a set of polynomials, then S_e are the irreducible factors of

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Lemma

If the singularities of F are cointained in S, then the singularities of $\int_0^\infty F \mathrm{d}\alpha_e$ are contained in S_e.

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This gives only very coarse upper bounds. For example, $z\bar{z}-1$ is spurious: It drops out in $S_{2,3}\cap S_{3,2} = \{z,\bar{z},1-z,1-\bar{z},z-\bar{z}\}$ because

$$S_{2,3} = \{z,\bar{z},1-z,1-\bar{z},z-\bar{z},z\bar{z}-z-\bar{z}\}\,.$$

Improvements

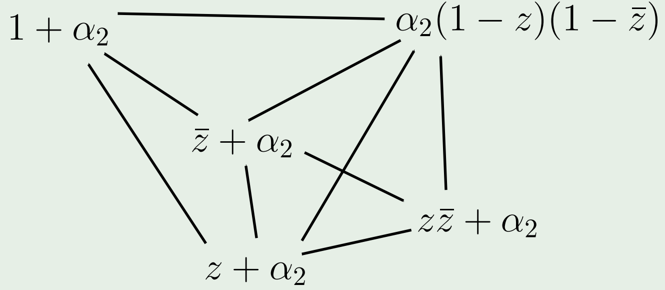
- Fubini: intersect over different orders
- Compatibility graphs

Compatibility graphs

Keep track of compatibilities $C \subset \binom{S}{2}$ between polynomials

- for S_e, we only need the resultants $[f,g]_e$ for compatible $\{f,g\} \in C$
- only pairs $[f,g]_e$, $[g,h]_e$ (with one polynomial in common) become compatible in C_e

Example (triangle)



Since $z\bar{z}\alpha_1 + \alpha_2$ and $\alpha_1 + \alpha_2$ are not compatible, their resultant $1 - z\bar{z}$ does not occur.

HyperInt: massive triangle

Graph polynomials:

```
> E:=[[2,3],[1,3],[1,2]]:
> M:=[[1,s1],[2,s2],[3,s3]],[m1^2,m2^2,m3^3]:
> psi:=graphPolynomial(E):
> phi:=secondPolynomial(E,M):
```

Polynomial reduction:

```
> L[{}]:=[{psi,phi}, {{psi,phi}}]:
> cgReduction(L, {s1,s2,s3,m1,m2,m3}, 2):
> L[{x[1],x[2]}][1];
```

$$\left\{ s_i + (m_j \pm m_k)^2, \sum_i s_i^2 - \sum_{i \neq j} s_i s_j \right\}$$

$$\cup \left\{ s_1 s_2 s_3 - \sum_i s_i^2 m_i^2 + \sum_i s_i (m_i^2 - m_j^2)(m_i^2 - m_k^2) + \sum_{i<j} s_i s_j (m_i^2 + m_j^2) \right\}$$

Linear reducibility

Definition

If for some order of variables (edges), all $S_{1,\ldots,k}$ are linear in α_{k+1}, then S (the Feynman graph G with $S = \{\psi, \varphi\}$) is called linearly reducible.

Lemma

If S is linearly reducible, the integral $\prod_e \int_0^\infty \mathrm{d}\alpha_e \ f$ of any rational function f with singularities in S is a MPL with symbol letters in $S_{1,\ldots,N}$.

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Integration of linearly reducible integrands in terms of MPL can be automatized via hyperlogarithms.

Definition (Hyperlogarithms, Lappo-Danilevsky 1927)

$$G(\sigma_1, \dots, \sigma_w; z) := \int_0^z \frac{\mathrm{d}z_1}{z_1 - \sigma_1} G(\sigma_2, \dots, \sigma_w; z_1)$$

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Symbolic integration: Rewriting hyperlogarithms

Problem: Given $G(\vec{\sigma}(\alpha); z)$, write it as hyperlogarithm with constant letters and final argument α. Recursive solution via

$$\mathrm{d}\, G(\vec{\sigma}; z) = \sum_{i=1}^{n} G(\cdots, \not{\sigma}_i, \cdots; z)\, \mathrm{d} \log \frac{\sigma_i - \sigma_{i-1}}{\sigma_i - \sigma_{i+1}} \qquad \sigma_0 := z, \sigma_{n+1} := 0$$

Example

$$\frac{\partial}{\partial \alpha} G\left(0, -\alpha; 1\right) = -\frac{1}{\alpha} G(-\alpha; 1)$$

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Example

$$\frac{\partial}{\partial \alpha} G\,(0, -\alpha; 1) = -\frac{1}{\alpha}\,[G(0; \alpha) - G(-1; \alpha)]$$

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$$\frac{\partial}{\partial \alpha} G(0, -\alpha; 1) = -\frac{1}{\alpha} [G(0; \alpha) - G(-1; \alpha)]$$
$$\Rightarrow G(0, -\alpha; 1) = -G(0, 0; \alpha) + G(0, -1; \alpha)$$

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Example

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Computation of integration constants relies on shuffle algebra

$$G(\vec{\sigma}; z) \cdot G(\vec{\tau}; z) = G(\vec{\sigma} \sqcup\!\sqcup\, \vec{\tau}; z).$$

All this is implemented in the Maple program `HyperInt`. Similar things can be done with other programs.

HyperInt

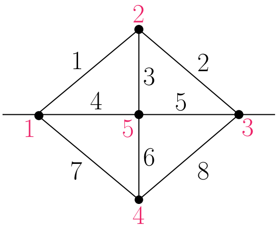
- open source Maple program
- integration of hyperlogarithms
- transformations of MPL to $G(\cdots;z)$
  Note: No further simplification (e.g. rewrite as $\mathrm{Li}_{2,2}$) provided!
- polynomial reduction
- graph polynomials
- symbolic computation of constants (no numerics)

Example

```
> read "HyperInt.mpl":
> hyperInt(polylog(2,-x)*polylog(3,-1/x)/x,x=0..infinity):
> fibrationBasis(%);
```

$$\frac{8}{7}\zeta_2^3$$

computes $\int_0^\infty \mathrm{Li}_2(-x)\,\mathrm{Li}_3(-1/x)\mathrm{d}x = \frac{8}{7}\zeta_2^3$.




```
> E := [[2,1],[2,3],[2,5],[5,1],[5,3],[5,4],[4,1],[4,3]]:
> psi := graphPolynomial(E):
> phi := secondPolynomial(E, [[1,1], [3,1]]):
> add((epsilon*log(psi^5/phi^4))^n/n!,n=0..2)/psi^2:
> hyperInt(eval(%,x[8]=1), [seq(x[n],n=1..7)]):
> collect(fibrationBasis(%), epsilon);
```

$$\left(254\zeta_7 + 780\zeta_5 - 200\zeta_2\zeta_5 - 196\zeta_3^2 + 80\zeta_2^3 - \frac{168}{5}\zeta_2^2\zeta_3\right)\varepsilon^2 + \left(-28\zeta_3^2 + 140\zeta_5 + \frac{80}{7}\zeta_2^3\right)\varepsilon + 20\zeta_5$$

HyperInt: triangle

Graph polynomials:

```
> E:=[[1,2],[2,3],[3,1]]:
> M:=[[3,1],[1,z*zz],[2,(1-z)*(1-zz)]]:
> psi:=graphPolynomial(E):
> phi:=secondPolynomial(E,M):
```

Integration:

```
> hyperInt(eval(1/psi/phi,x[3]=1),[x[1],x[2]]):
> factor(fibrationBasis(%,[z,zz]));
```

$$(\mathrm{Hlog}\,(1;z)\,\mathrm{Hlog}\,(0;zz) - \mathrm{Hlog}\,(0;z)\,\mathrm{Hlog}\,(1;zz) + \mathrm{Hlog}\,(0,1;zz) - \mathrm{Hlog}\,(1,0;zz) + \mathrm{Hlog}\,(1,0;z) - \mathrm{Hlog}\,(0,1;z))/(z-zz)$$

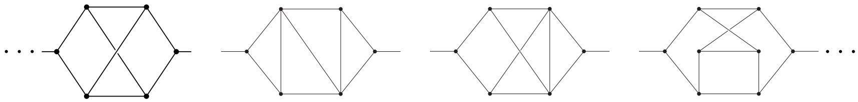
Polynomial reduction:

```
> L[{}]:=[{psi,phi}, {{psi,phi}}]:
> cgReduction(L):
> L[{x[1],x[2]}][1];
```

$$\{-1+z, -1+zz, -zz+z\}$$

Linearly reducible families (fixed loop order)

1. all $\leq$ 4 loop massless propagators (Panzer)



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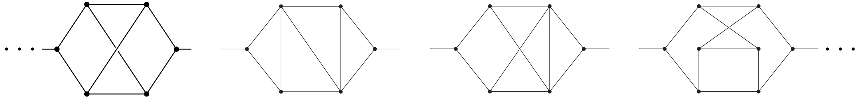
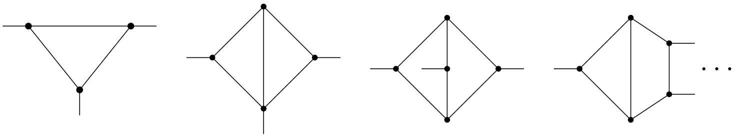


2. all ≤ 3 loop massless off-shell 3-point (Chavez & Duhr, Panzer)



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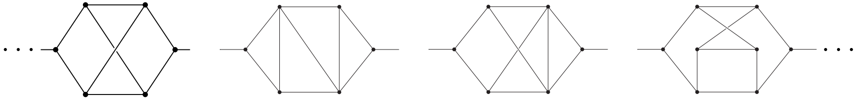

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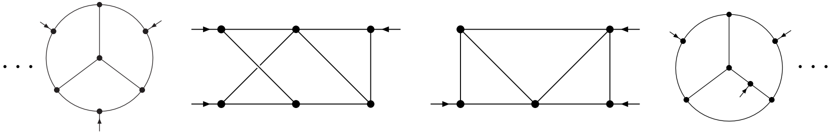

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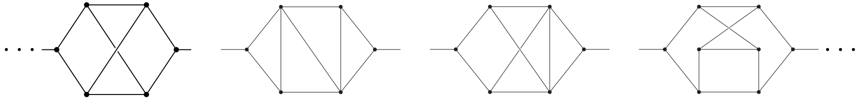

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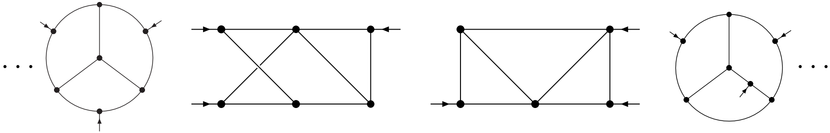
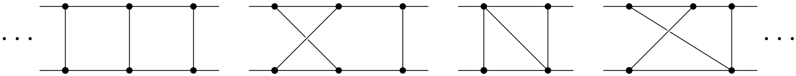
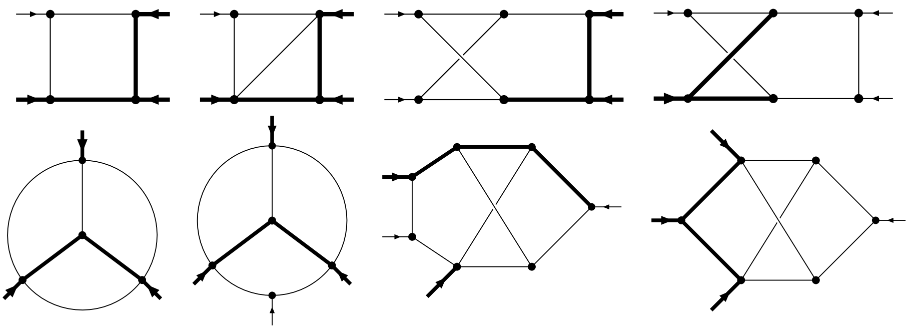


3. all ≤ 2 loop massless on-shell 4-point (Lüders)

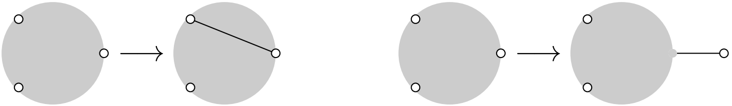


Linearly reducible massive graphs (some examples)

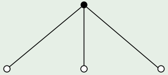


Linearly reducible families (infinite)

- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]



Example



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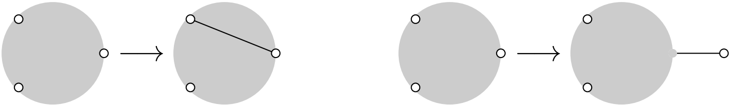

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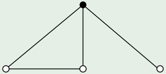

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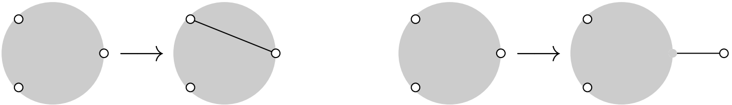

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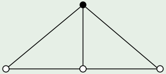

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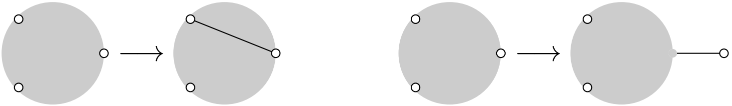

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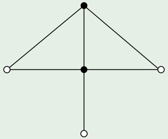

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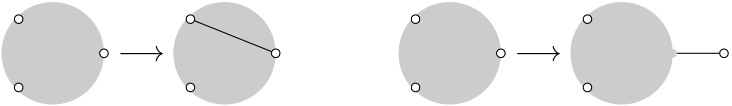

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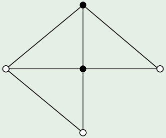

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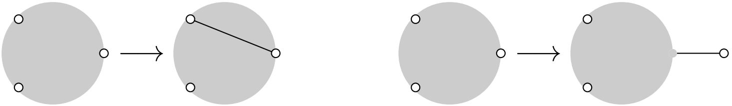

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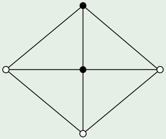

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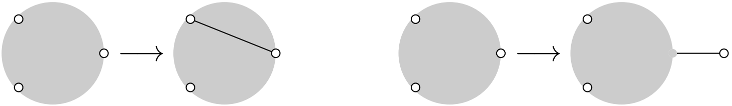

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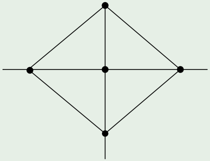

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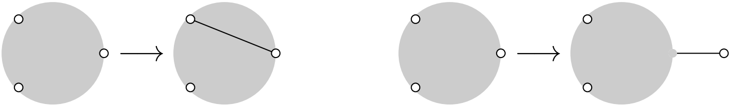

Theorem (Panzer)

All ϵ-coefficients of these graphs (off-shell) are MPL over the alphabet $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, 1-z\bar{z}, 1-z-\bar{z}, z\bar{z}-z-\bar{z}\}$.

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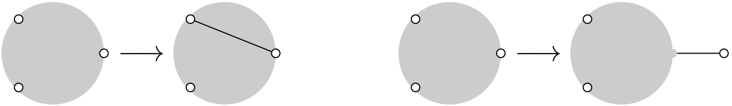

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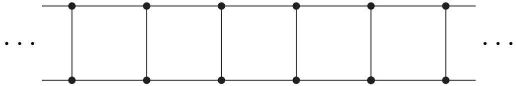
- minors of ladder-boxes (up to 2 legs off-shell)



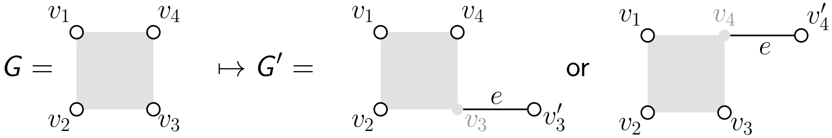
Theorem (Panzer)

All ϵ-coefficients of these graphs are MPL. For the massless case, the alphabet is just $\{x, 1+x\}$ *for* $x = s/t$.

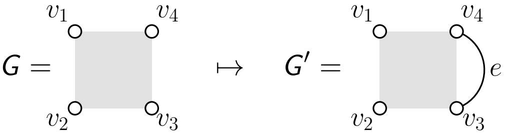
4-point recursions

Start with the box and repeat, in any order:

- Appending a vertex:



- Adding an edge:



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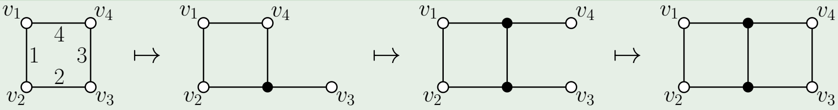
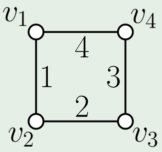


Forest polynomials

Let f_3, f_4, f_{12} and f_{14} denote the spanning forest polynomials such that

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \qquad \begin{aligned} f_{12} &= \alpha_2\alpha_4 \\ f_{14} &= \alpha_1\alpha_3 \end{aligned} \qquad \begin{aligned} f_3 &= \alpha_2\alpha_3 \\ f_4 &= \alpha_3\alpha_4 \end{aligned}$$

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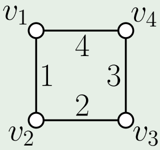
Definition

$$F(G; z) := \int_{\mathbb{R}_+^E} \psi_G^{-D/2} \cdot \delta^{(4)}\left(\frac{f}{\psi} - z\right) \prod_{e \in E} \mathrm{d}\alpha_e^{a_e - 1} \alpha_e \qquad (\mathbb{R}_+^4 \longrightarrow \mathbb{R}_+)$$

Example



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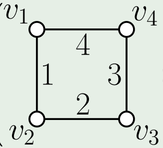
$$F(G; z) := \int_{\mathbb{R}_+^E} \psi_G^{-D/2} \cdot \delta^{(4)}\left(\frac{f}{\psi} - z\right) \prod_{e \in E} \mathrm{d}\alpha_e^{a_e - 1} \alpha_e \qquad (\mathbb{R}_+^4 \longrightarrow \mathbb{R}_+)$$

Example



$$F\left(G \,; z \right) = \begin{cases} \dfrac{1}{z_3 z_4} & (D = 4) \\[2ex] \dfrac{z_{12}}{\big[\underbrace{z_{12}(z_{14} + z_3 + z_4) + z_3 z_4}_{Q}\big]^2} & (D = 6) \end{cases}$$

Forest polynomials

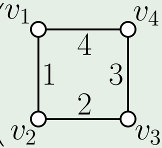
Let f_3, f_4, f_{12} and f_{14} denote the spanning forest polynomials such that

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

Definition

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \frac{F(G;z)\,\Omega}{\left[(p_1+p_2)^2 z_{12} + (p_1+p_4)^2 z_{14} + p_3^2 z_3 + p_4^2 z_4\right]^{\mathrm{sdd}}}$$

Example

$F\Bigg($


$; z\Bigg) =$

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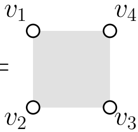
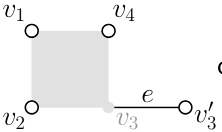
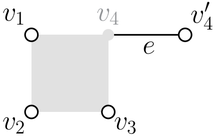
Appending a vertex

$G =$ 

$\mapsto G' =$ 

or 

Using $(f'_{12}, f'_{14}, f'_3, f'_4, \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$,

$$F(G'; z) = \int_0^{z_4} F(G; z_{12}, z_{14}, z_3, z_4 - \alpha_e)\ \alpha_e^{a_e - 1} \mathrm{d}\alpha_e$$

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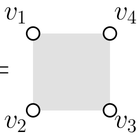
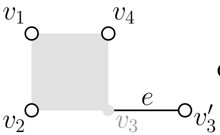
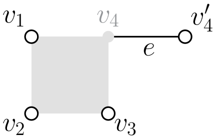
$G =$ 

$\mapsto G' =$ 

or 

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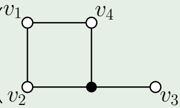
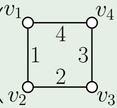
$F\Bigg($ 

$; z\Bigg) = \int_0^{z_3} F\Bigg($ 

$; z_{12}, z_{14}, z'_3, z_4\Bigg)\ \mathrm{d}z'_3$

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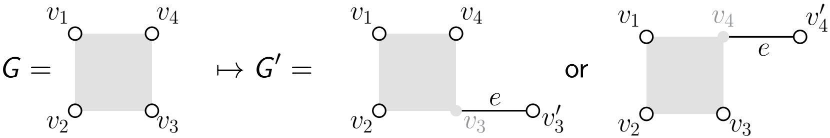

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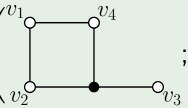
$F\Bigg($  $; z\Bigg)$

$$= \int_0^{z_3} \frac{z_{12}\ \mathrm{d}z_3'}{[z_{12}\,(z_{14} + z_3' + z_4) + z_3' z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$$

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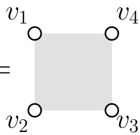
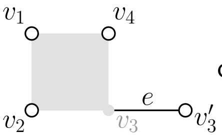
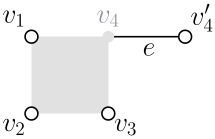
$G =$ 

$\mapsto G' =$ 

or 

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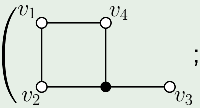
$F\Big($ 

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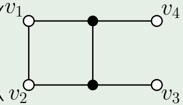
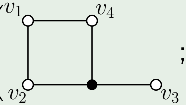
$F\Big($ 

$; z\Big) = \int_0^{z_4} F\Big($ 

$; z_{12}, z_{14}, z_3, z_4'\Big)\ \mathrm{d}z_4'$

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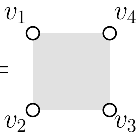
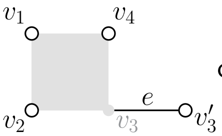
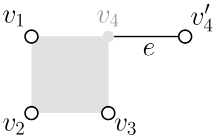
$G =$ 

$\mapsto G' =$ 

or 

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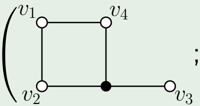
$F\Bigg($ 

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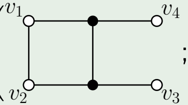
$F\Bigg($ 

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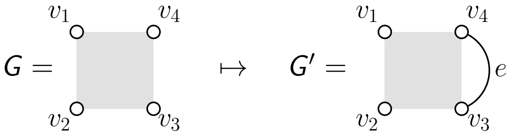
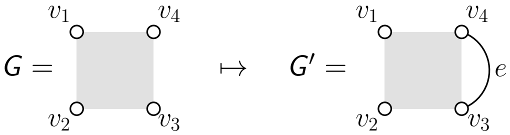
Adding an edge



$$F_{G'}(z) = Q^{a_e + \mathsf{sdd} - D} \int_0^{z_{12}} x^{D/2-2} \left[Q^{D/2 - \mathsf{sdd}} \cdot F_G \right]_{z_{12} = z_{12} - x} \mathrm{d}x$$

Adding an edge



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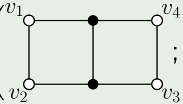
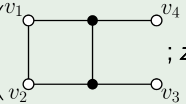
$F\Big($  $; z\Big) = \dfrac{1}{Q^2}\displaystyle\int_0^{z_{12}} F\Big($  $; z_{12} - x, z_{14}, z_3, z_4\Big)\, x\,\mathrm{d}x$

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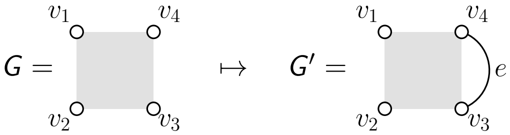

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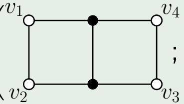
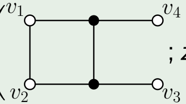
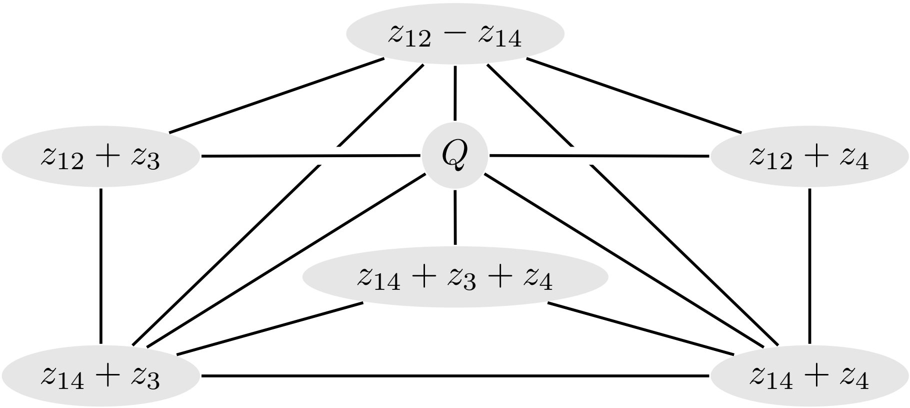
$F\Big($  $; z\Big) = \dfrac{1}{Q^2}\displaystyle\int_0^{z_{12}} F\Big($  $; z_{12}-x, z_{14}, z_3, z_4\Big)\, x\,\mathrm{d}x$

$$= \frac{z_{12}-z_{14}}{Q^2}\left[\ln\frac{Q}{z_3 z_4}\ln\frac{(z_{14}+z_3)(z_{14}+z_4)}{z_{14}(z_{14}+z_3+z_4)} - \mathrm{Li}_2\left(\frac{z_3 z_4(z_{14}-z_{12})}{z_{14}Q}\right)\right]$$

$$+\frac{z_{12}-z_{14}}{Q^2}\mathrm{Li}_2\left(\frac{z_3 z_4}{Q}\right)+\frac{z_{12}}{Q^2}\ln\frac{z_{14}z_3 z_4}{z_{12}(z_{14}+z_3)(z_{14}+z_4)}-\frac{\ln(z_3 z_4/Q)}{Q(z_{14}+z_3+z_4)}$$

Compatibility graph of box-ladders



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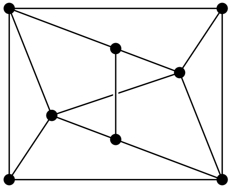
Thank you!

Massless ϕ^4 theory: primitive sixth roots of unity

$P_{7,11} =$  is not linearly reducible!

Tenth denominator:

$$\begin{aligned} d_{10} = {} & \alpha_2\alpha_4^2\alpha_1 + \alpha_2\alpha_4^2\alpha_3 - \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_2^2\alpha_4\alpha_1 + \alpha_2^2\alpha_4\alpha_3 \\ & - 2\alpha_2\alpha_3^2\alpha_4 - \alpha_2^2\alpha_3^2 - 2\alpha_2^2\alpha_3\alpha_1 - 2\alpha_2\alpha_3^2\alpha_1 - \alpha_3^2\alpha_4^2 \\ & - 2\alpha_3^2\alpha_4\alpha_1 - \alpha_2^2\alpha_1^2 - 2\alpha_2\alpha_3\alpha_1^2 - \alpha_3^2\alpha_1^2. \end{aligned}$$

Change variables: $\alpha_3 = \frac{\alpha_3'\alpha_1}{\alpha_1+\alpha_2+\alpha_4}$, $\alpha_4 = \alpha_4'(\alpha_2 + \alpha_3')$ and $\alpha_1 = \alpha_1'\alpha_4'$,

$$d_{10}' = (\alpha_2 + \alpha_3')(\alpha_2 + \alpha_2\alpha_4' - \alpha_1')(\alpha_1'\alpha_4' + \alpha_2 + \alpha_2\alpha_4' + \alpha_3'\alpha_4')$$

Final result: *not a multiple zeta value*, instead MPL at $e^{i\pi/3}$

$\sqrt{3}\,\mathcal{P}(P_{7,11})$

$$= \operatorname{Im}\left(\tfrac{19\,285}{6}\zeta_9 \operatorname{Li}_2 - \tfrac{1029}{2}\zeta_7 \operatorname{Li}_4 + 240\zeta_3^2(9\operatorname{Li}_{2,3} - 7\zeta_3 \operatorname{Li}_2)\right) - \tfrac{93\,824}{9675}\pi^3\zeta_{3,5}$$

$$+ \tfrac{2592}{215}\operatorname{Im}\Big(36\operatorname{Li}_{2,2,2,5} + 27\operatorname{Li}_{2,2,3,4} + 9\operatorname{Li}_{2,2,4,3} + 9\operatorname{Li}_{2,3,2,4} + 3\operatorname{Li}_{2,3,3,3}$$

$$- 43\zeta_3(\operatorname{Li}_{2,3,3} + 3\operatorname{Li}_{2,2,4})\Big) - \tfrac{96\,393\,596\,519\,864\,341\,538\,701\,979}{790\,371\,465\,315\,684\,594\,157\,620\,000}\pi^{11}$$

$$+ \tfrac{216}{14\,755\,731\,798\,995}\operatorname{Im}\Big(2\,539\,186\,130\,125\,890\operatorname{Li}_8\zeta_3 - 1\,269\,593\,065\,062\,945\operatorname{Li}_{2,9}$$

$$-413\,965\,317\,054\,502\operatorname{Li}_6\zeta_5 - 996\,412\,983\,391\,539\operatorname{Li}_{3,8}$$

$$-546\,306\,741\,059\,841\operatorname{Li}_{4,7} - 156\,228\,639\,992\,955\operatorname{Li}_{5,6}\Big)$$

$$+ \tfrac{2592}{10\,945\,435}\pi^2\operatorname{Im}\Big(287\,205\operatorname{Li}_{2,7} - 574\,410\operatorname{Li}_6\zeta_3 + 55\,687\operatorname{Li}_{4,5} + 168\,941\operatorname{Li}_{3,6}\Big)$$

$$+ \pi\left(\tfrac{11\,613\,751}{9030}\zeta_5^2 + \tfrac{267\,067}{602}\zeta_{3,7} - \tfrac{31\,104}{215}\operatorname{Re}(3\operatorname{Li}_{4,6} + 10\operatorname{Li}_{3,7})\right)$$

Abbreviation: $\operatorname{Li}_{n_1,\dots,n_r} := \operatorname{Li}_{n_1,\dots,n_r}(e^{i\pi/3})$