A Basis of Finite Feynman Integrals

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based on work with Erik Panzer and Robert M. Schabinger

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consider \( L \) loop Euclidean Feynman integrals:

\[
I = \int \frac{d^d k_1}{i\pi^{d/2}} \cdots \frac{d^d k_L}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \cdots D_N^{a_N}}
\]

where \( a_i \in \mathbb{Z} \) and e.g. \( D_1 = k_1^2 - m_1^2 \) etc.

linear dependencies:
- integration-by-parts (IBP) identities [Tkachov, Chetyrkin '81]
- systematic reduction to small number of master integrals [Laporta '00]
- think of it as linear vector space with some arbitrary basis (master integrals)

expansion in \( \epsilon = (4 - d)/2 \)
- typically sufficient for phenomenological applications
- Laurent coefficients are simpler integrals

solving methods typically based on
- direct integration of Feynman (Schwinger) parameter integrals
- differential equations
An improved basis for Feynman parameters

consider *Feynman parameter representation* of multi-loop integral

\[
I = \frac{\Gamma\left(\nu - \frac{Ld}{2}\right)(-1)^\nu}{\prod_{i=1}^{N} \Gamma(\nu_i)} \left[ \prod_{j=1}^{N} \int_{0}^{\infty} dx_j \right] \delta(1 - x_N) U^{\nu - (L+1)d/2} F^{-\nu + Ld/2} \prod_{k=1}^{N} x_k^{\nu_k - 1}
\]

where \( \nu = \sum_i \nu_i \), \( \nu_i \) denotes propagator multiplicity

presence of *subdivergencies* (= divergencies from Feynman parameter integrations) implies:

- can’t directly expand in \( \epsilon \)
- no straight-forward analytical or numerical integration

generic approaches to *singularity resolution*:

1. sector decomposition [Binoth, Heinrich ‘00]
2. polynomial exponent raising [Tkachov ‘96, Passarino ‘00]
3. regularising dimension shifts [Panzer ‘14]
4. basis of finite Feynman integrals (“dims & dots”) [AvM, Schabinger, Panzer ‘14]
Sector decomposition: shortcomings

calculate to $O(\epsilon)$:

$$I(\epsilon) = \int_0^1 dt \ t^{-1-\epsilon}(1 - t)^{-1-2\epsilon} \ _2F_1(\epsilon, 1 - \epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$:

$$I_1(\epsilon) = \int_0^{1/2} dt \ t^{-1-\epsilon}(1 - t)^{-1-2\epsilon} \ _2F_1(\epsilon, 1 - \epsilon; -\epsilon; t)$$

$$I_2(\epsilon) = \int_{1/2}^1 dt \ t^{-1-\epsilon}(1 - t)^{-1-2\epsilon} \ _2F_1(\epsilon, 1 - \epsilon; -\epsilon; t).$$

rescale, expand in plus distributions, evaluate:

$$I_1(\epsilon) = -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3} \pi^2 - 8 \ln(2)\right) \epsilon + O(\epsilon^2)$$

$$I_2(\epsilon) = \frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3} \pi^2 + 8 \ln(2)\right) \epsilon + O(\epsilon^2).$$

result:

$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3} \pi^2\right) \epsilon + O(\epsilon^2).$$

note:

- split up of domain introduces spurious terms $\ln(2)$
- spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility & prevents analytical integration a la [Brown '08; Panzer '14]
An example for subdivergencies

\[ \int \frac{d^dk_1}{i\pi^{d/2}} \int \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{((k_1 + k_2)^2 - m^2) k_1^2 k_2^2} \]

\[ = -\Gamma(-1 + 2\epsilon) \int_0^\infty dx_1 \delta(1 - x_1) \int_0^\infty dx_2 \int_0^\infty dx_3 \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{1-2\epsilon}, \]

with Symanzik polynomials

\[ \mathcal{U} = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U}. \]

- **can't expand integrand in \( \epsilon \):**

\[ \int \frac{d^dk}{i\pi^{d/2}} \]

\[ = - (m^2)^{1-2\epsilon} \frac{\Gamma(-1 + 2\epsilon)\Gamma(\epsilon)\Gamma(1 - \epsilon)}{1 - \epsilon} \]

\( \Gamma(\epsilon) \) signals subdivergence

- **Euclidean** integrals: all divergencies from integration **boundaries**

- notation here: restrict to one or several parameters approaching **zero** (not infinity)
Systematic recognition of subdivergencies

- follow [Panzer '14]
- consider subsets

\[ \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1\}, \{x_2\}, \{x_3\} \]

- for each subset \( J \) consider scaling with \( \lambda \):

\[ J \rightarrow \lambda J \]

for integrand \( P \equiv \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{-1-2\epsilon} \):

\[ P \rightarrow P_{J,\lambda} = \lambda^{\deg_J(P)} \tilde{P} \quad \text{where} \quad \lim_{\lambda \rightarrow 0} \tilde{P} = O(\lambda^0) \]

and the integral measure

\[ \prod_{i=1}^{3} dx_i \rightarrow \lambda^{|J|} \prod_{i=1}^{3} dx_i \]

and read off:

**convergence index**

\[ \omega_J(P) = |J| + \deg_J(P), \]

\[ \lim_{\epsilon \rightarrow 0} \omega_J(P) \leq 0 \quad \Leftrightarrow \quad \text{presence of non-integrable subdivergence} \]
Panzer’s regularising shift

integrand can be regularized by dimension-shifts [Panzer ’14]:

1. pick \( J \) for which \( \lim_{\epsilon \to 0} \omega_J(P) \leq 0 \)
2. multiply by \( 1 = \int_0^\infty d\lambda \, \delta(\lambda - x_J) \) with \( x_J = \sum_{j \in J} x_j \)
3. rescale \( x_j \to \lambda x_j \) for all \( j \in J \) and perform partial integration (surface term vanishes)
4. new integrand

\[
P' = -\frac{1}{\omega_J(P)} \frac{\partial}{\partial \lambda} \tilde{P} \bigg|_{\lambda \to 1}.
\]

has improved convergence by design

5. iterate until no subdivergencies
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5. has improved convergence by design
6. iterate until no subdivergencies

applicability in practice:

- problem: proliferation of terms
- solution: integration by parts (IBP) reductions
Our proposal: minimal dims & dots

decompose wrt quasi-finite basis

\[
\frac{4(1 - \epsilon)(3 - 4\epsilon)(1 - 4\epsilon)}{\epsilon s^2}
\]

\[-\frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2}\]

\[-\frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1 - 2\epsilon)\epsilon^3 s^3}\]

basis consists of standard Feynman integrals, but

- in shifted dimensions
- with additional dots (propagators taken to higher powers)
- old reg. shifts generated $\mathcal{O}(10\text{MB})$, here: 3 lines! (more severe at higher loops)
**Existence of quasi-finite basis**

1. start with some basis $B$ for topology and subtopologies
2. assume master $b$ not quasi-finite and has integrand
   
   $$P = \mathcal{U}^{\nu-(L+1)d/2} \mathcal{F}^{-\nu+Ld/2} \prod_{j=1}^{N} x_j^{\nu_j-1}, \quad \text{where } \nu = \sum_{i=1}^{N} \nu_i$$

3. consider regularizing dimension shift:
   
   $$P' = -\frac{1}{\omega_J(P)} \prod_{j=1}^{N} x_j^{\nu_j-1} \left\{ \left( \nu - \frac{(L+1)d}{2} \right) \mathcal{U}^{\nu+L-(L+1)(d+2)/2} \mathcal{F}^{-(\nu+L)+L(d+2)/2} \frac{\partial \tilde{U}}{\partial \lambda} \bigg|_{\lambda \rightarrow 1} \right.$$  

   $$\left. + \mathcal{F} \text{ derivative term} \right\},$$

   with $\mathcal{U}_{J,\lambda} = \lambda^{\deg_J(\mathcal{U})} \tilde{U}$

4. picking any monomial from $\frac{\partial \tilde{U}}{\partial \lambda} \bigg|_{\lambda \rightarrow 1}$ or $\frac{\partial \tilde{F}}{\partial \lambda} \bigg|_{\lambda \rightarrow 1}$ gives
   
   **dimension-shifted** and **dotted** integral with **improved convergence** !

5. choose one term such that new integral $b'$ is independent of $B \setminus b$
6. replace $b \rightarrow b'$ and iterate until $B$ free of subdivergences (quasi-finite)
7. optional: transition quasi-finite $\rightarrow$ finite integrals
Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

**Algorithm: construction of (quasi-)finite basis**

- systematic scan for (quasi-)finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change
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**Remarks:**

- computationally expensive part shifted to IBP solver (Fire, Reduze, LiteRed)
- efficient, easy to automate (implemented in dev. version of Reduze 2)
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon
**Example 1a: Non-planar Two-loop Vertex (Quasi-finite)**

\[
(4 - 2\epsilon) = (4 - 2\epsilon),
\]

\[
(4 - 2\epsilon) = \frac{2 - 3\epsilon}{\epsilon},
\]

\[
(4 - 2\epsilon) = \frac{4(1 - \epsilon)(3 - 4\epsilon)(1 - 4\epsilon)}{\epsilon s^2},
\]

\[
- \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2},
\]

\[
- \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1 - 2\epsilon)\epsilon^3 s^3}.
\]
Example 1b: Non-planar Two-loop Vertex (Finite)

\[ (4 - 2\epsilon) \]
\[ = -\frac{4s^2}{\epsilon(1 - 2\epsilon)} \]
\[ (8 - 2\epsilon) \]

\[ (4 - 2\epsilon) \]
\[ = \frac{2(2 - 3\epsilon)s}{\epsilon^2} \]
\[ (8 - 2\epsilon) \]

\[ (4 - 2\epsilon) \]
\[ = \frac{4(1 - \epsilon)(1 - 4\epsilon)}{\epsilon s} \]
\[ (8 - 2\epsilon) \]

\[ - \frac{2(2 - 3\epsilon)(5 - 21\epsilon + 14\epsilon^2)}{\epsilon^4 s} \]
\[ (8 - 2\epsilon) \]

\[ + \frac{4(2 - 3\epsilon)(7 - 31\epsilon + 26\epsilon^2)}{\epsilon^4(1 - 2\epsilon)s} \]
\[ (8 - 2\epsilon) \]
Example 2: massless planar double box family

\[ b_1 = (6 - 2\epsilon) \]

\[ b_2 = (6 - 2\epsilon) \]

\[ b_3 = (6 - 2\epsilon) \]

\[ b_4 = (6 - 2\epsilon) \]

\[ b_5 = (6 - 2\epsilon) \]

\[ b_6 = (4 - 2\epsilon) \]

\[ b_7 = (4 - 2\epsilon) \]

\[ b_8 = (6 - 2\epsilon) \]
**Example 3: Three-loop form factor**

- massless quark and gluon form factors:
  - simplest objects to study IR properties of QCD

- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] ⇐ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factor @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]
\[ F_3^q = \frac{1}{\epsilon^6} \left\{ \begin{array}{c} (10-2\epsilon) \\ c_1 + c_2 + c_3 + c_4 + c_5 \\ (10-2\epsilon) \\ +c_6 + c_7 + c_8 + c_9 + c_{10} + c_{11} + c_{12} + c_{13} + c_{14} \\ (8-2\epsilon) \\ +c_{15} + c_{16} + c_{17} + c_{18} + c_{19} + c_{20} + c_{21} + c_{22} \end{array} \right\} \]
Example 4: Four-loop form factor [AvM, Panzer, Schabinger; in progress]

- example: a non-planar 12-line top level topology @ 4-loops

- analytical result with HypInt [Panzer]:

\[
(6-2\epsilon) = \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \mathcal{O}(\epsilon) \approx 3.18074 + \mathcal{O}(\epsilon)
\]

- numerical result with Fiesta [A. Smirnov]: 3.18082 + \epsilon 58.8288 + \mathcal{O}(\epsilon^2)
Numerical evaluations

Advantages of (quasi-)finite basis:

- Straight-forward to integrate numerically (in principle)
- No blow up in number of numerical integrations (speed, stability)
- No cancellation of spurious structures (stability)

Experiments with numerical evaluations:

- Naive straight-forward implementation works already quite well
- Convenient: employ existing sector decomposition programs
  Fiesta, SecDec and sector_decomposition
- (Quasi-)finite integrals: faster & more reliable
Conclusions

basis of finite integrals (dims and dots):

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: quasi-finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations

results and outlook:

- massless form factors @ 3-loops: first independent rederivation at higher weights
- massless form factors @ 4-loops: cusp anomalous dimension and more (in progress)
- phase space integrals (outlook)