A Basis of Finite Feynman Integrals

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based on work with Erik Panzer and Robert M. Schabinger

arXiv:1411.7392 + to appear

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Multi-loop Feynman integrals

consider L **loop Euclidean Feynman integrals**:

$$I = \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \cdots \frac{\mathrm{d}^d k_L}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \cdots D_N^{a_N}}$$

where $a_i \in \mathbb{Z}$ and e.g. $D_1 = k_1^2 - m_1^2$ etc.

linear dependencies:

- integration-by-parts (IBP) identities [Tkachov, Chetyrkin '81]
- systematic reduction to small number of master integrals [Laporta '00]
- think of it as linear vector space with some arbitrary basis (master integrals)

expansion in $\epsilon = (4 - d)/2$

- typically sufficient for phenomenological applications
- Laurent coefficients are simpler integrals

solving methods typically based on

1. direct integration of Feynman (Schwinger) parameter integrals
2. differential equations

An improved basis for Feynman parameters

consider **Feynman parameter representation** of multi-loop integral

$$I = \frac{\Gamma\left(\nu - \frac{Ld}{2}\right)(-1)^{\nu}}{\prod_{i=1}^{N}\Gamma(\nu_i)} \left[\prod_{j=1}^{N}\int_0^{\infty} \mathrm{d}x_j\right] \delta(1-x_N)\, \mathcal{U}^{\nu-(L+1)d/2}\mathcal{F}^{-\nu+Ld/2}\prod_{k=1}^{N} x_k^{\nu_k - 1}$$

where $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity

presence of **subdivergencies** (= divergencies from Feynman parameter integrations) implies:

- can't directly expand in ϵ
- no straight-forward analytical or numerical integration

generic approaches to **singularity resolution**:

1. sector decomposition [Binoth, Heinrich '00]
2. polynomial exponent raising [Tkachov '96, Passarino '00]
3. regularising dimension shifts [Panzer '14]
4. New **basis of finite Feynman integrals** ("dims & dots") [AvM, Schabinger, Panzer '14]

Sector decomposition: shortcomings

calculate to $\mathcal{O}(\epsilon)$:

$$I(\epsilon) = \int_0^1 \mathrm{d}t\; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$:

$$I_1(\epsilon) = \int_0^{1/2} \mathrm{d}t\; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$
$$I_2(\epsilon) = \int_{1/2}^{1} \mathrm{d}t\; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)\,.$$

rescale, expand in plus distributions, evaluate:

$$I_1(\epsilon) = -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3}\pi^2 - 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$
$$I_2(\epsilon) = -\frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3}\pi^2 + 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

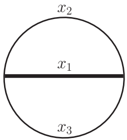
result:

$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3}\pi^2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

note:

- split up of domain introduces **spurious terms** $\ln(2)$
- spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility & prevents **analytical integration** a la [Brown '08; Panzer '14]

An example for subdivergencies



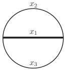
$$= \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \int \frac{\mathrm{d}^d k_2}{i\pi^{d/2}} \frac{1}{((k_1+k_2)^2 - m^2)\, k_1^2 k_2^2}$$

$$= -\Gamma(-1+2\epsilon) \int_0^\infty \mathrm{d}x_1 \delta(1-x_1) \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{1-2\epsilon}\,,$$

with Symanzik polynomials

$$\mathcal{U} = x_1x_2 + x_1x_3 + x_2x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U}\,.$$

- **can't expand integrand in ϵ:**



$$= -\left(m^2\right)^{1-2\epsilon} \frac{\Gamma(-1+2\epsilon)\Gamma(\epsilon)\Gamma(1-\epsilon)}{1-\epsilon}$$

$\Gamma(\epsilon)$ signals subdivergence

- **Euclidean** integrals: all divergencies from integration **boundaries**
- notation here: restrict to one or several parameters approaching **zero** (not infinity)

Systematic recognition of subdivergencies

- follow [Panzer '14]
- consider subsets

$$\{x_1, x_2\}, \quad \{x_1, x_3\}, \quad \{x_2, x_3\}, \quad \{x_1\}, \quad \{x_2\}, \quad \{x_3\}$$

- for each subset J consider **scaling with λ**:

$$J \to \lambda J$$

for integrand $P \equiv \mathcal{U}^{-3+3\epsilon}\mathcal{F}^{1-2\epsilon}$:

$$P \to P_{J_\lambda} = \lambda^{\deg_J(P)}\tilde{P} \qquad \text{where } \lim_{\lambda\to 0}\tilde{P} = \mathcal{O}(\lambda^0)$$

and the integral measure

$$\prod_{i=1}^{3} \mathrm{d}x_i \to \lambda^{|J|} \prod_{i=1}^{3} \mathrm{d}x_i$$

and read off:

convergence index

$$\omega_J(P) = |J| + \deg_J(P)\,,$$

$$\lim_{\epsilon\to 0}\omega_J(P) \le 0 \quad \Leftrightarrow \quad \text{presence of non-integrable subdivergence}$$

Panzer's regularising shift

integrand can be regularized by dimension-shifts [Panzer '14]:

1. pick J for which $\lim_{\epsilon\to 0}\omega_J(P) \leq 0$
2. multiply by $1 = \int_0^\infty \mathrm{d}\lambda\ \delta(\lambda - x_J)$ with $x_J = \sum_{j\in J} x_j$
3. rescale $x_j \to \lambda x_j$ for all $j \in J$ and perform partial integration (surface term vanishes)
4. new integrand
$$P' = -\frac{1}{\omega_J(P)}\frac{\partial}{\partial\lambda}\tilde{P}\bigg|_{\lambda\to 1}.$$
has improved convergence by design
5. iterate until no subdivergencies

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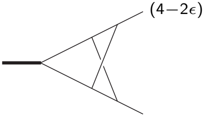
applicability in practice:

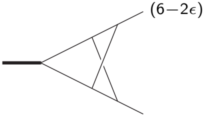
- problem: proliferation of terms
- solution: integration by parts (IBP) reductions

Our proposal: minimal dims & dots

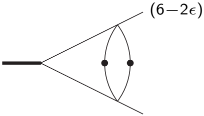
decompose wrt **quasi-finite basis**



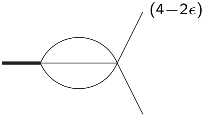
$$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$$



$$- \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2}$$



$$- \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3}$$



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basis consists of standard Feynman integrals, but

- in **shifted dimensions**
- with additional **dots** (propagators taken to higher powers)
- old reg. shifts generated $\mathcal{O}(10\mathrm{MB})$, here: 3 lines ! (more severe at higher loops)

Existence of quasi-finite basis

1. start with some basis B for topology and subtopologies
2. assume master b not quasi-finite and has integrand

$$P = \mathcal{U}^{\nu-(L+1)d/2}\mathcal{F}^{-\nu+Ld/2}\prod_{j=1}^{N} x_j^{\nu_j-1}, \qquad \text{where } \nu = \sum_{i=1}^{N}\nu_i$$

3. consider regularizating dimension shift:

$$P' = -\frac{1}{\omega_J(P)}\prod_{j=1}^{N} x_j^{\nu_j-1}\Big\{\Big(\nu - \tfrac{(L+1)d}{2}\Big)\mathcal{U}^{(\nu+L)-(L+1)(d+2)/2}\mathcal{F}^{-(\nu+L)+L(d+2)/2}\frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\Big|_{\lambda\to 1} + \mathcal{F}\text{ derivative term}\Big\},$$

   with $\mathcal{U}_{J_\lambda} = \lambda^{\deg_J(\mathcal{U})}\widetilde{\mathcal{U}}$

4. picking any monomial from $\frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\Big|_{\lambda\to 1}$ or $\frac{\partial\widetilde{\mathcal{F}}}{\partial\lambda}\Big|_{\lambda\to 1}$ gives

   dimension-shifted and **dotted** integral with **improved convergence** !

5. choose one term such that new integral b' is independent of $B \setminus b$
6. replace $b \to b'$ and iterate until B free of subdivergences (quasi-finite)
7. optional: transition quasi-finite $\to$ finite integrals

Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

Algorithm: construction of (quasi-)finite basis

- systematic scan for (quasi-)finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

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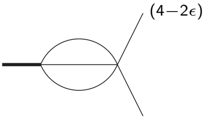
remarks:

- computationally expensive part shifted to IBP solver (`Fire`, `Reduze`, `LiteRed`)
- efficient, easy to automate (implemented in dev. version of `Reduze` 2)
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

Example 1a: non-planar two-loop vertex (quasi-finite)



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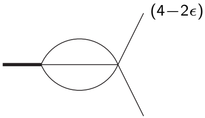

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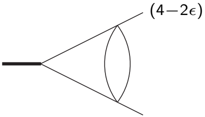
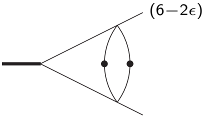


$$= \frac{2-3\epsilon}{\epsilon}$$



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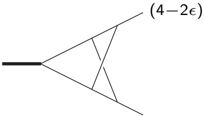
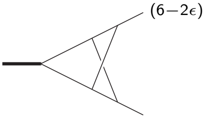
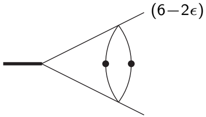


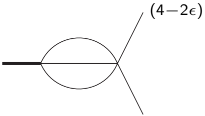
$$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$$



$$- \frac{10-65\epsilon+131\epsilon^2-74\epsilon^3}{\epsilon^3 s^2}$$



$$- \frac{14-119\epsilon+355\epsilon^2-420\epsilon^3+172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3}$$



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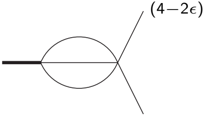
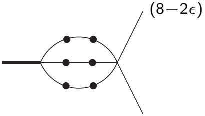
Example 1b: non-planar two-loop vertex (finite)



$$= -\frac{4s^2}{\epsilon(1-2\epsilon)}$$



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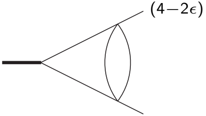
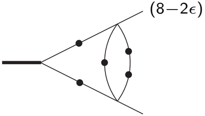


$$= \frac{2(2-3\epsilon)s}{\epsilon^2}$$



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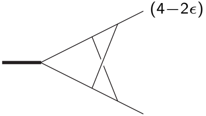
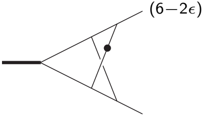
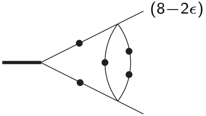


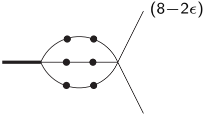
$$= -\frac{4(1-\epsilon)(1-4\epsilon)}{\epsilon s}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4 s}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)s}$$



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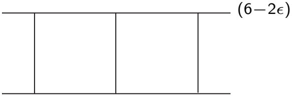
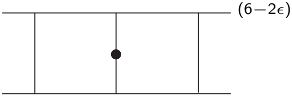
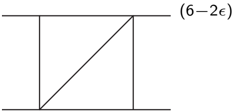
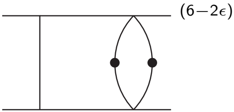
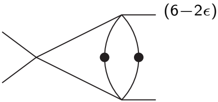
Example 2: massless planar double box family

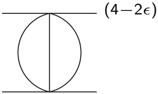
$b_1 =$ 

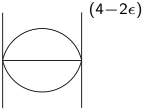
$b_2 =$ 

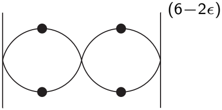
$b_3 =$ 

$b_4 =$ 

$b_5 =$ 

$b_6 =$ 

$b_7 =$ 

$b_8 =$ 

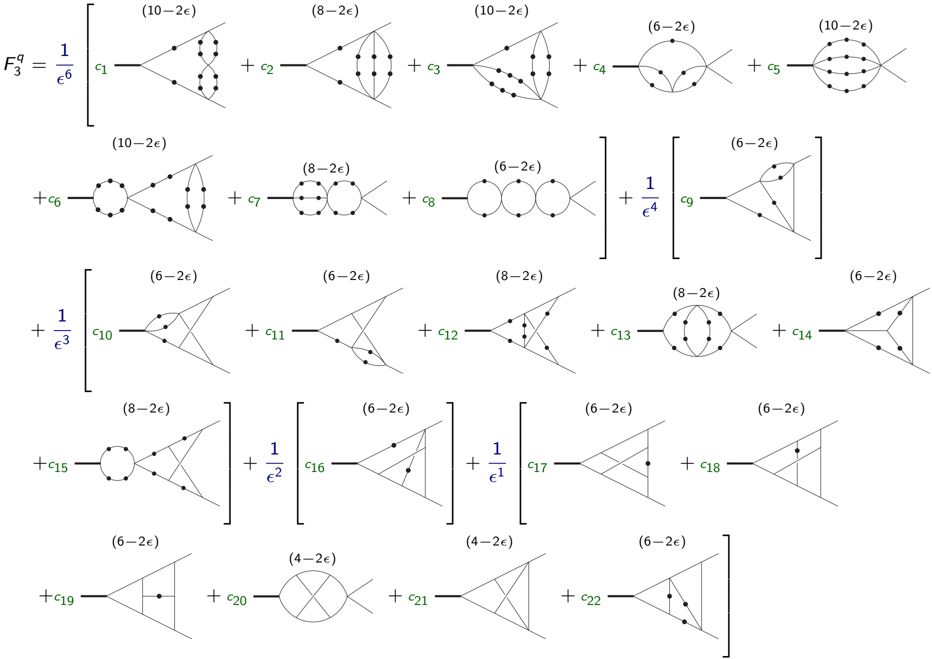
Example 3: Three-loop form factor

- massless quark and gluon form factors:
  - simplest objects to study IR properties of QCD

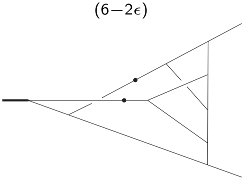
- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] $\Leftarrow$ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factor @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]

Example 3: Three-loop form factor [AvM, Panzer, Schabinger; to appear]



- example: a non-planar 12-line top level topology @ 4-loops
- analytical result with HypInt [Panzer]:



$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \mathcal{O}(\epsilon) \qquad \approx 3.18074 + \mathcal{O}(\epsilon)$$

- numerical result with Fiesta [A. Smirnov]: $3.18082 + \epsilon\, 58.8288 + \mathcal{O}(\epsilon^2)$

Numerical evaluations

advantages of (quasi-)finite basis:

- straight-forward to integrate numerically (in principle)
- no blow up in number of numerical integrations (speed, stability)
- no cancellation of spurious structures (stability)

experiments with numerical evaluations:

- naive straight-forward implementation works already quite well
- convenient: employ existing sector decomposition programs `Fiesta`, `SecDec` and `sector_decomposition`
- (quasi-)finite integrals: faster & more reliable

Conclusions

basis of finite integrals (dims and dots):

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: quasi-finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations

results and outlook:

- massless form factors @ 3-loops: first independent rederivation at heigher weights
- massless form factors @ 4-loops: cusp anomalous dimension and more (in progress)
- phase space integrals (outlook)