

THRESHOLD LOGARITHMS BEYOND LEADING POWER

Lorenzo Magnea

University of Torino - INFN Torino

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Outline

- Introduction
- Threshold resummations at leading power
- Gathering evidence beyond leading power
- Next-to-eikonal approximation
- Solving a collinear conundrum
- Outlook

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- Introduction
- Threshold resummations at leading power
- Gathering evidence beyond leading power
- Next-to-eikonal approximation
- Solving a collinear conundrum ← **For the full story, see next talk by Domenico Bonocore**
- Outlook

INCEPTION



Logarithms

📌 **Multi-scale** problems in **renormalizable** quantum field theories have perturbative corrections of the form $\alpha_s^n \log^k (Q_i^2/Q_j^2)$, which may **spoil** the reliability of the perturbative expansion. However, they **carry important physical information**.

- **Renormalization** and **factorization** logs: $\alpha_s^n \log^n (Q^2/\mu^2)$
- **High-energy logs**: $\alpha_s^n \log^{n-1} (s/t)$
- **Sudakov** logs: $\alpha_s^n \log^{2n-1} (1-z)$, $1-z = W^2/Q^2, 1-M^2/\hat{s}, Q_\perp^2/Q^2, \dots$

📌 **Logarithms** encode **process-independent** features of perturbation theory. For **Sudakov** logs: the structure of **infrared** and **collinear** divergences.

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \implies \ln(m^2/Q^2)$$

- For **inclusive** observables: **analytic** resummation to high logarithmic accuracy.
- For **exclusive** final states: **parton shower** event generators, (N)LL accuracy. 

Gradually merging

📌 **Resummation** probes the **all-order structure** of perturbation theory.

- **Non-perturbative** contributions to QCD cross sections can be estimated.
- Links to the **strong coupling** regime can be established for special gauge theories.

More logarithms

- **Threshold logarithms** are associated with kinematic variables ξ that **vanish** at **Born level** and get **corrections** that are **enhanced** because **phase space** for real radiation is **restricted** near **partonic** threshold: examples are $1 - T$, $1 - M^2/\hat{s}$, $1 - x_{BJ}$.
- At **leading power** in the threshold variable ξ logarithms are **directly related** to **soft** and **collinear divergences**: real radiation is proportional to factors of

$$\frac{1}{\xi^{1+p\epsilon}} = -\frac{1}{p\epsilon} \delta(\xi) + \left(\frac{1}{\xi}\right)_+ - p\epsilon \left(\frac{\log \xi}{\xi}\right)_+ + \dots$$

Cancels virtual IR poles

Leading power threshold logs

- **Beyond** the **leading power**, $1/\xi$, the perturbative cross section takes the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[c_{nm}^{(-1)} \left(\frac{\log^m \xi}{\xi}\right)_+ + c_n^{(\delta)} \delta(\xi) + c_{nm}^{(0)} \log^m \xi + \dots \right]$$

Resummed to high accuracy

All-order structure in some cases

NLP threshold logs

- The **structure** of **NLP** threshold logarithms may be understood to **all orders**.

LEADING POWER



Electroweak annihilation

We will **focus** on processes involving **parton annihilation** into **electroweak final states** (**Drell-Yan, Higgs, di-boson** final states): very well **understood** at **LP**, **simpler** at **NLP**.

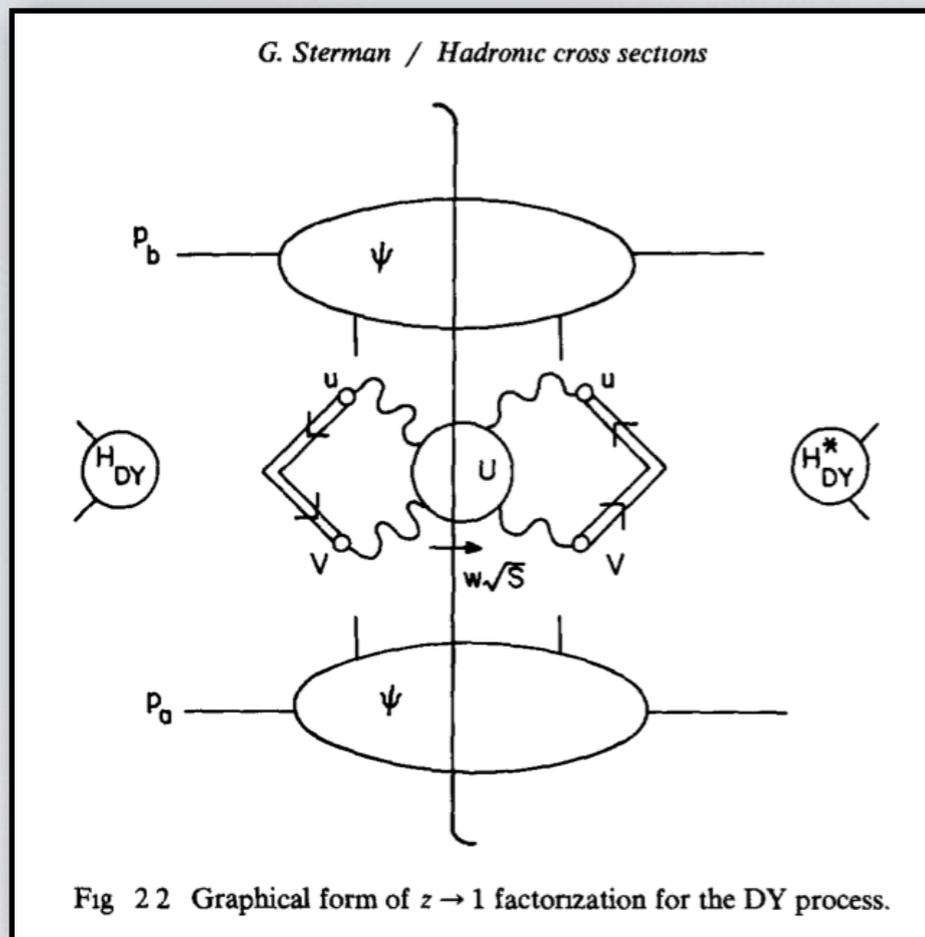
- **LP** threshold **resummation** is based on **factorization**: the **Mellin-space** partonic cross section reads

$$\omega(N, \epsilon) = |H_{\text{DY}}|^2 \psi(N, \epsilon)^2 U(N) + \mathcal{O}\left(\frac{1}{N}\right).$$

- **Collinear** poles can be **subtracted** with suitable parton distributions,

$$\hat{\omega}_{\overline{\text{MS}}}(N) \equiv \frac{\omega(N, \epsilon)}{\phi_{\overline{\text{MS}}}(N, \epsilon)^2}$$

- Each factor in ω obeys **evolution equations** near **threshold**, leading to **exponentiation**.



The original factorization near threshold

- **Real** and **virtual** contributions can be treated **separately**.

$$\psi_R(N, \epsilon) = \exp \left\{ \int_0^1 dz \frac{z^{N-1}}{1-z} \int_z^1 \frac{dy}{1-y} \kappa_\psi(\bar{\alpha}((1-y)^2 Q^2), \epsilon) \right\}.$$

The perturbative exponent

A classic way to **organize** Sudakov logarithms is in terms of the **Mellin (Laplace) transform** of the momentum space cross section (**Catani et al. 93**),

$$\begin{aligned} d\sigma(\alpha_s, N) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{k=0}^{2n} c_{nk} \log^k N + \mathcal{O}(1/N) \\ &= H(\alpha_s) \exp \left[\log N g_1(\alpha_s \log N) + g_2(\alpha_s \log N) + \alpha_s g_3(\alpha_s \log N) + \dots \right] + \mathcal{O}(1/N) \end{aligned}$$

This displays the main **features of Sudakov resummation**

- 🔊 **Predictive:** a **k**-loop calculation determines **g_k** and thus a whole **tower** of logarithms to all orders in perturbation theory.
- 🔊 **Effective:**
 - the **range of applicability** of perturbation theory is **extended** (finite order: **α_s log²N** small. NLL resummed: **α_s** small);
 - the renormalization **scale dependence** is naturally **reduced**.
- 🔊 **Theoretically interesting:** resummation **ambiguities** related to the **Landau pole** give access to non-perturbative **power-suppressed corrections**.
- 🔊 **Well understood:**
 - **NLL** Sudakov resummations **exist** for most **inclusive** observables at hadron colliders, **NNLL** and approximate **N³LL** in simple cases.

Color singlet hard scattering

A well-established formalism exists for **distributions** in processes that are **electroweak at tree level** (Gardi, Grunberg 07). For an observable r **vanishing in the two-jet limit**

$$\frac{d\sigma}{dr} = \delta(r) [1 + \mathcal{O}(\alpha_s)] + C_R \frac{\alpha_s}{\pi} \left\{ \left[-\frac{\log r}{r} + \frac{b_1 - d_1}{r} \right]_+ + \mathcal{O}(r^0) \right\} + \mathcal{O}(\alpha_s^2)$$

The Mellin (Laplace) transform, $\sigma(N) = \int_0^1 dr (1-r)^{N-1} \frac{d\sigma}{dr}$

exhibits **log N** singularities that can be organized in **exponential form**

$$\sigma(\alpha_s, N, Q^2) = H(\alpha_s) \mathcal{S}(\alpha_s, N, Q^2) + \mathcal{O}(1/N)$$

where the exponent of the **'Sudakov factor'** is in turn a Mellin transform

$$\mathcal{S}(\alpha_s, N, Q^2) = \exp \left\{ \int_0^1 \frac{dr}{r} \left[(1-r)^{N-1} - 1 \right] \mathcal{E}(\alpha_s, r, Q^2) \right\}$$

and the general form of the **kernel** is

$$\mathcal{E}(\alpha_s, r, Q^2) = \int_{r^2 Q^2}^{rQ^2} \frac{d\xi^2}{\xi^2} A(\alpha_s(\xi^2)) + B(\alpha_s(rQ^2)) + D(\alpha_s(r^2 Q^2))$$

where **A** is the **cusp** anomalous dimension, and **B** and **D** have **distinct physical characters**.

Non-logarithms

Delta-function terms arise from **virtual** corrections and **phase space** integration. They yield **constants** in Mellin space (“ π^2 ”) which can be controlled and “**exponentiate**” for simple processes (Parisi 80; Sterman 87; Eynck, Laenen, LM 03; Ahrens, Becher, Neubert, Yang 08),

- For **EW annihilation**, virtual terms reconstruct the **full form factor**.
- In **dimensional regularization**, each term exponentiates with **no prefactor**.

$$\begin{aligned}\omega(N, \epsilon) &= |H_{\text{DY}} \mathcal{R}(\epsilon) \sqrt{U_V(\epsilon)}|^2 \psi_R(N, \epsilon)^2 U_R(N) + \mathcal{O}\left(\frac{1}{N}\right) \\ &= |\Gamma(Q^2, \epsilon)|^2 \psi_R(N, \epsilon)^2 U_R(N, \epsilon) + \mathcal{O}\left(\frac{1}{N}\right),\end{aligned}$$

- **Real** and **virtual** factors are **separately** finite.

$$\hat{\omega}_{\overline{\text{MS}}}(N) \equiv \frac{\omega(N, \epsilon)}{\phi_{\overline{\text{MS}}}(N, \epsilon)^2} = \left(\frac{|\Gamma(Q^2, \epsilon)|^2}{\phi_V(\epsilon)^2}\right) \left(\frac{\psi_R(N, \epsilon)^2 U_R(N, \epsilon)}{\phi_R(N, \epsilon)^2}\right) + \mathcal{O}\left(\frac{1}{N}\right)$$

- An **improved** resummation formula can be written for **DY, DIS** and **Higgs total rates**: **all** constants are **defined** in the exponent.

$$\begin{aligned}\hat{\omega}_{\overline{\text{MS}}}(N) &= \left|\frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)}\right|^2 \left(\frac{\Gamma(-Q^2, \epsilon)}{\phi_V(\epsilon)}\right)^2 \exp\left[F_{\overline{\text{MS}}}(\alpha_s)\right] \times \\ &\times \exp\left[\int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left\{ 2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) + \right. \right. \\ &\left. \left. + D(\alpha_s((1-z)^2 Q^2)) \right\}\right] + \mathcal{O}\left(\frac{1}{N}\right).\end{aligned}$$

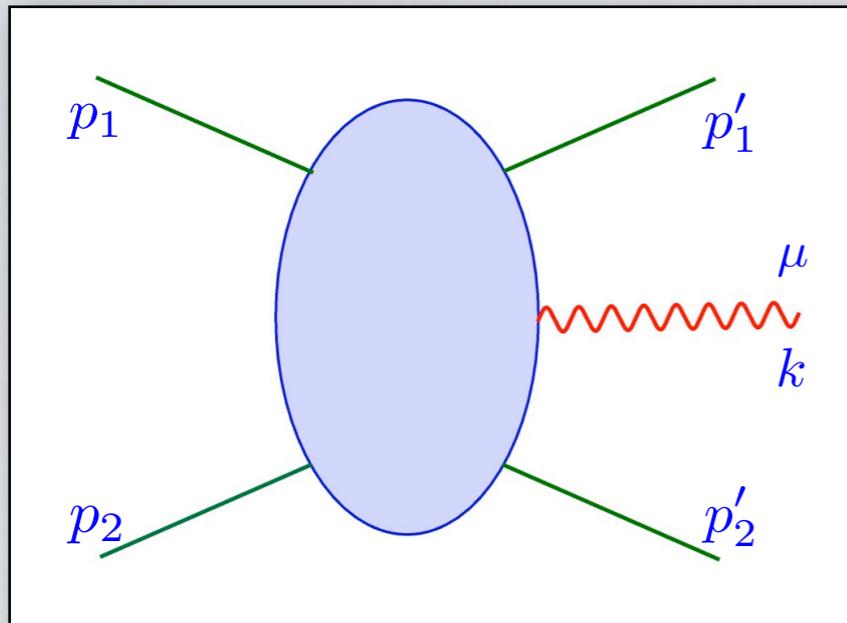
Less predictive than conventional resummation: the exponent receives corrections **order by order**. **Empirically**, exponentiated **lower-order** constants provide **much** of the exact result.

GATHERING EVIDENCE



The LBKD Theorem

The **earliest evidence** that infrared effects can be **controlled** at **NLP** is **Low's theorem** (Low 58)



A radiative matrix element

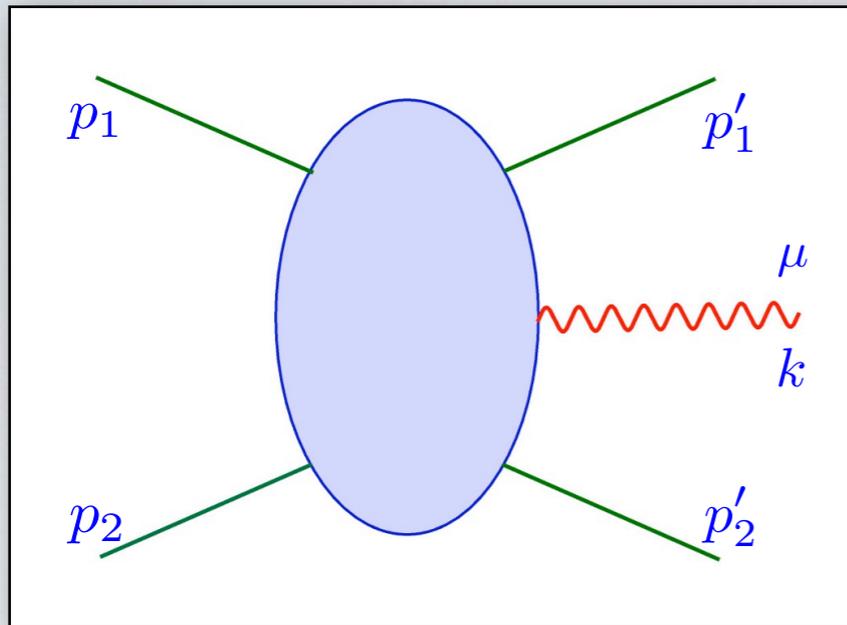
$$M_\mu = e \left(\frac{p_{1\mu}'}{p_1' \cdot k} - \frac{p_{1\mu}}{p_1 \cdot k} \right) T(\nu, \Delta) \quad (1.7)$$

$$+ e \left(\frac{p_{1\mu}' p_{2\mu}' \cdot k}{p_1' \cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_{2\mu} \cdot k}{p_1 \cdot k} - p_{2\mu} \right) \frac{\partial T(\nu, \Delta)}{\partial \nu} + O(k),$$

Low's original expression for the radiative matrix element

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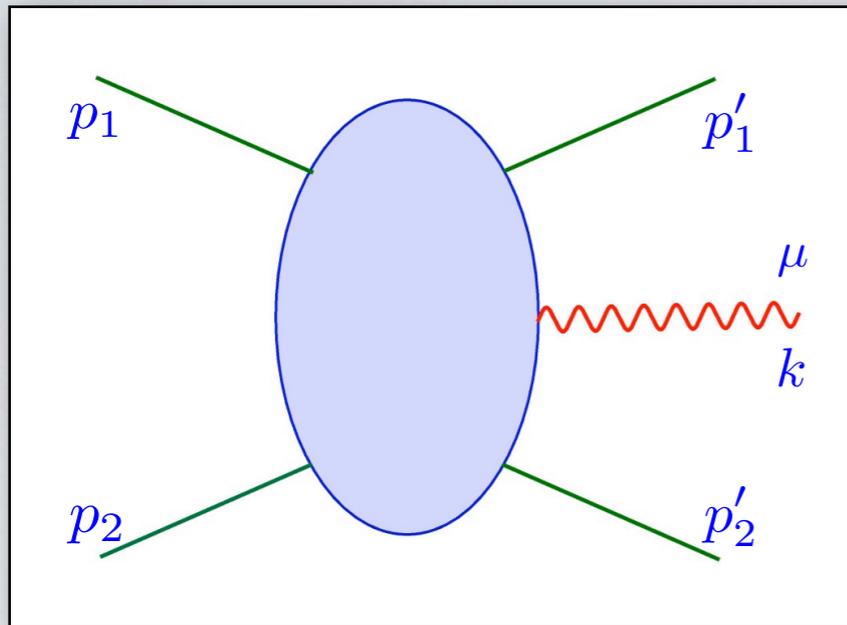
Eikonal approximation

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Next-to-eikonal contribution

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Eikonal approximation

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Next-to-eikonal contribution

The **radiative matrix element** for the emission of a (next-to-) soft photon is **determined** by the **Born amplitude** T and its **first derivative** w.r.t. external momenta.

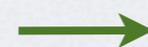
- Low's result established for a **single charged scalar** particle, follows from **gauge invariance**.
- It **generalizes** the well known properties of soft emissions in the **eikonal approximation**.
- The theorem was **extended** by (Burnett, Kroll 68) to particles with **spin**.
- The **LBK** theorem applies to **massive particles** and uses the **mass** as a **collinear cutoff**.
- It was **extended** to **massless** particles by (Del Duca 90), as discussed **below**.

Modified DGLAP

An **important source** of known **NLP** logarithms is the **DGLAP anomalous dimension**. Non-trivial **connections** between **LP** and **NLP** logarithms in **DGLAP** were **uncovered** (**Moch, Vermaseren, Vogt 08**) and made **systematic** (**Dokshitzer, Marchesini, Salam 08**).

Conventional DGLAP for a quark distribution reads

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x, \mu^2) = \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}, \mu^2\right) P_{qq}(z, \alpha_s(\mu^2)).$$



$$\mu^2 \frac{\partial}{\partial \mu^2} \tilde{q}(N, \mu^2) = \gamma_N(\alpha_s(\mu^2)) \tilde{q}(N, \mu^2),$$

The **large-N** behavior of the anomalous dimension is **single-logarithmic** in the MS scheme. **NLP** terms **suppressed** by **N** are **related** to **LP**

$$\begin{aligned} \gamma_N(\alpha_s) = & -A(\alpha_s) \ln \bar{N} + B_\delta(\alpha_s) \\ & - C_\gamma(\alpha_s) \frac{\ln \bar{N}}{N} + D_\gamma(\alpha_s) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right), \end{aligned}$$

MVV relations

$$C_1 = 0, \quad C_2 = 4C_F A_1, \quad C_3 = 8C_F A_2. \quad (3.12)$$

Especially the relation for C_3 is very suggestive and seems to call for a structural explanation.

These relation **extend** to the function **D**, and recursively **to all orders: modified** splitting functions can be **defined** which **vanish** at **large x beyond one loop**.

Modified DGLAP

DMS, with refinements implied by (Basso, Korchemsky 06) propose to **modify DGLAP** as

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi\left(\frac{x}{z}, z^\sigma \mu^2\right) \mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right).$$

applying to **both PDF's** and **fragmentation**, with $\sigma = \pm 1$ respectively, and the **same kernel** (Gribov-Lipatov reciprocity). The resulting kernel \mathcal{P} and is claimed to **vanish** as $z \rightarrow 1$ **beyond one loop** in the “physical” MC scheme where $\alpha_s = \gamma_{\text{cusp}}$. Therefore

$$\mathcal{P}(z, \alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s)\delta(1-z) + \mathcal{O}(1-z).$$

The modified equation **cannot be diagonalized** by Mellin transform: it must be solved **by iteration**, using a formal translation operator

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} e^{-\ln z (\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \sigma \frac{\partial}{\partial \ln \mu^2})} \times \psi\left(\frac{x}{z}, \mu^2\right) \mathcal{P}(z, \alpha_s(\mu^2)),$$

In practice, this procedure **constructs** a **modified kernel** where high-order terms are **generated by shifts** of lower-orders

An educated guess

Available **NLP information** can be combined in an **ansatz** for generalized threshold resummation applicable to **EW annihilation** processes and **DIS** (Laenen, LM, Stavenga 08).

$$\ln[\hat{\omega}(N)] = \mathcal{F}_{\text{DY}}(\alpha_s(Q^2))$$

$$+ \int_0^1 dz z^{N-1} \left\{ \frac{1}{1-z} D \left[\alpha_s \left(\frac{(1-z)^2 Q^2}{z} \right) \right] \right.$$

$$\left. + 2 \int_{Q^2}^{(1-z)^2 Q^2/z} \frac{dq^2}{q^2} P_s[z, \alpha_s(q^2)] \right\}_+,$$

Exponentiation of constants

Refinement of phase space

DMS kernel

$$\frac{2}{1-z} \longrightarrow \frac{2z}{1-z}$$

This expression, and similar ones for **DIS** and **Higgs** production via gluon fusion, incorporate

- The **exponentiation** of **N-independent** terms.
- A treatment of **phase space** consistent up to $\mathcal{O}(1-z)$, including **running coupling** effects.
- The **DMS modification** of the **DGLAP** kernel, including the **NLP term** in the **LO kernel**.
- **Note:** **DMS** brings **to the exponent** a C_F^2 contribution **crucial** to **fit** two-loop **NLP** logs.

An educated guess: Drell-Yan

Parametrizing DY with

$$\hat{\omega}(N) = \sum_{i=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^n \left[\sum_{m=0}^{2n} a_{nm} \ln^m \bar{N} + \sum_{m=0}^{2n-1} b_{nm} \frac{\ln^m \bar{N}}{N} \right] + \mathcal{O} \left(\frac{\ln^p N}{N^2} \right),$$

Table 1

Comparison of exact and resummed 2-loop coefficients for the Drell-Yan cross section. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

	C_F^2	$C_A C_F$	$n_f C_F$
b_{23}	4	0	0
b_{22}	$\frac{7}{2}$	$\frac{11}{6}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$

Table 2

Comparison of exact and resummed 2-loop coefficients for the DIS structure function. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

	C_F^2	$C_A C_F$	$n_f C_F$
d_{23}	$\frac{1}{4}$	0	0
d_{22}	$\frac{39}{16}$	$\frac{11}{48}$	$-\frac{1}{24}$
d_{21}	$\frac{7}{4}\zeta_2 - \frac{49}{32}$	$-\frac{5}{4}\zeta_2 + \frac{1333}{288}$	$-\frac{1}{4}\zeta_2 + \frac{565}{288}$
d_{20}	$\frac{15}{4}\zeta_3 - \frac{47}{16}\zeta_2 - \frac{431}{64}$	$-\frac{11}{4}\zeta_3 + \frac{13}{48}\zeta_2 - \frac{17579}{1728}$	$\frac{5}{4}\zeta_3 + \frac{7}{16}\zeta_2 - \frac{953}{1728}$

- Only **one-loop NLP** and **DMS input** has been used in the resummation formula.
- Leading NLP** logarithms are **reproduced exactly** for **all color structures** at two loops.
- NLL** and **NNLL NLP** logarithms are **well approximated** but **not exact**.
- Similar** results hold for **three-loop DIS**, using **two-loop information** in the exponent.

Towards systematics

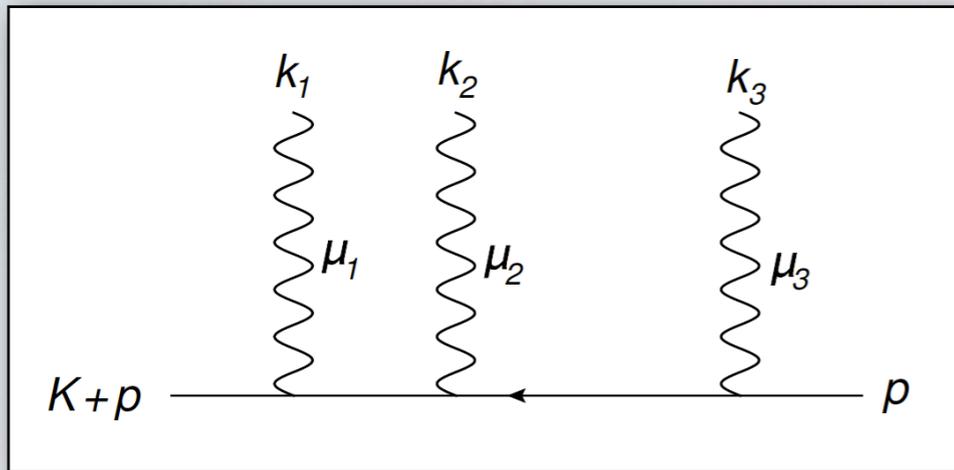
The problem of **NLP** threshold logarithms has been of interest for a **long time**, and several **different approaches** have been proposed. Recent years have seen a **resurgence of interest**, both from a **theoretical** point of view and for **phenomenology**.

- 📌 **Early attempts** include a study of the impact of **NLP** logs on the **Higgs** cross section by **Kraemer, Laenen, Spira (98)**; work on **F_L** by **Akhoury and Sterman (99)** (logs without plus distributions are however **leading**) and work by **Grunberg et al. (07-09)** on **DIS**.
- 📌 **Important results** can be obtained by using **physical kernels** (**Vogt et al. 09-14**) which are conjectured to be **single-logarithmic** at large **z** , which poses **constraints** on their **factorized** expression. Note in particular a **recent application** to **Higgs** production by **De Florian, Mazzitelli, Moch, Vogt (14)**.
- 📌 **Useful approximations** can be obtained by combining **constraints** from **large N** with **high-energy** constraints for **$N \sim 1$** and **analyticity** (**Ball, Bonvini, Forte, Marzani, Ridolfi, 13**), together with **phase space** refinements.
- 📌 **SCET techniques** can be applied and indeed may be **well-suited** to the problem: a thorough **one-loop** analysis was given in (**Larkoski, Neill, Stewart, 15**).
- 📌 A lot of recent **formal work** on the behavior of **gauge** and **gravity** scattering **amplitudes beyond the eikonal** limit was triggered by a link to **asymptotic symmetries** of the **S** matrix (**many authors from A(ndy Strominger) to Z(vi Bern), 14-15**).

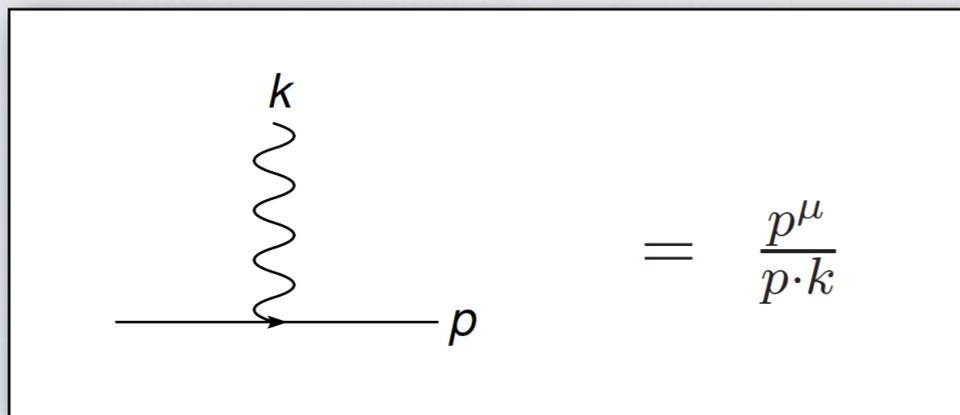
NEXT-TO-SOFT APPROXIMATION



On the eikonal approximation



A fast particle emitting soft photons



Eikonal Feynman rule

- Taking the **soft approximation** at **leading power** on emissions from an **energetic** (or very **massive**) particle yields a set of **simplified Feynman rules**.
- These rules correspond to emissions from a **Wilson line** oriented **along the trajectory** of the energetic particle, in the **same color** irrep.

$$\Phi_n(\lambda_2, \lambda_1) = P \exp \left[ig \int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A(\lambda n) \right] .$$

- The results **do not depend on the energy** and **spin** of the emitter, only on its **direction** and **color charge**.
- **Physically**, we are **neglecting** the **recoil** of the emitter: the only effect of interaction with soft radiation is that the **emitter acquires a phase**.
- The **soft** limit of a **multi-particle** amplitude is a **correlator of Wilson lines**

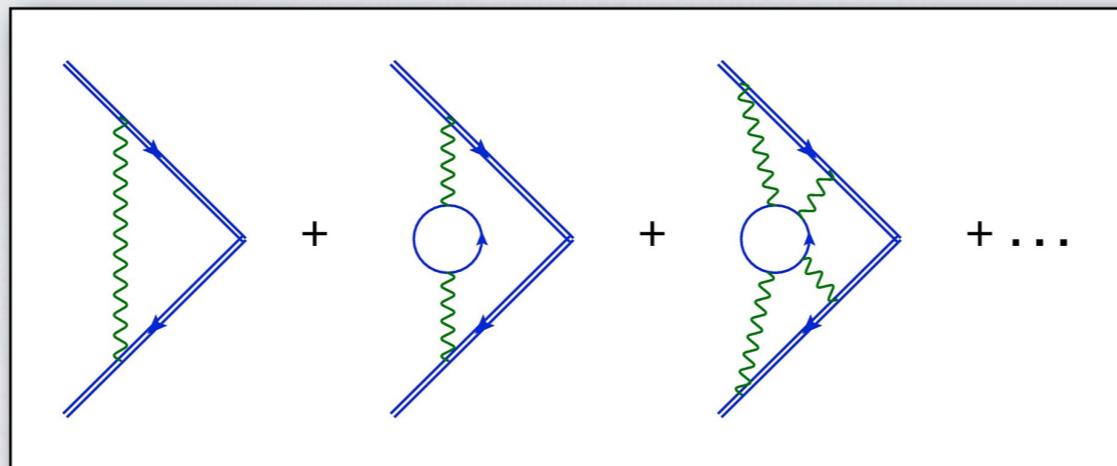
Infrared exponentiation

All correlators of Wilson lines, regardless of shape, resum in **exponential form**.

$$S_n \equiv \langle 0 | \Phi_1 \otimes \dots \otimes \Phi_n | 0 \rangle = \exp(\omega_n)$$

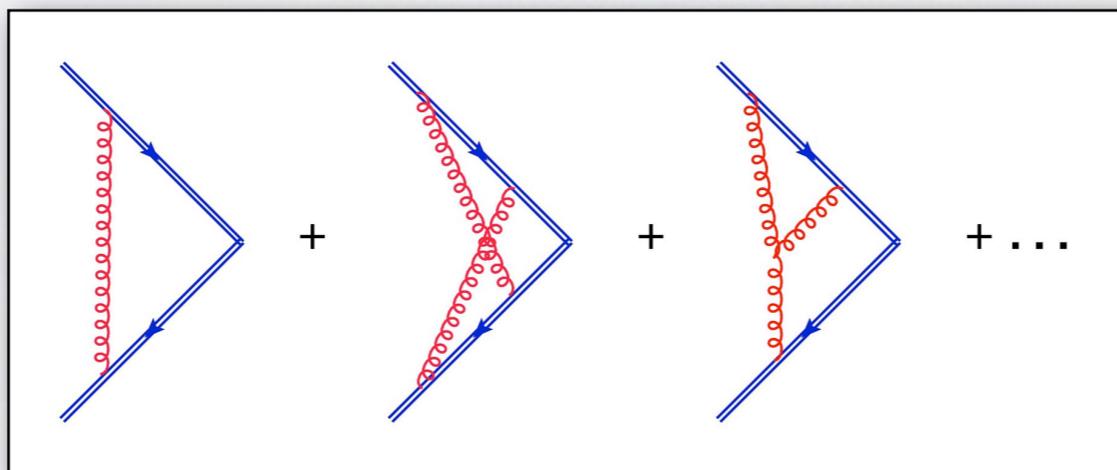
Diagrammatic rules exist to compute **directly the logarithm** of the correlators.

$$\omega_{2,\text{QED}} =$$



Only **connected** photon **subdiagrams** contribute to the logarithm.

$$\omega_{2,\text{QCD}} =$$



Only gluon **subdiagrams** which are **two-eikonal irreducible** contribute to the logarithm. They have **modified color factors**.

For **eikonal form factors**, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).

Multiparticle webs

The concept of **web** generalizes non-trivially to the case of **multiple Wilson lines**.
(Gardi, Smillie, White, et al).

A **web** is a **set of diagrams** which **differ** only by the **order** of the **gluon attachments** on each Wilson line. They are **weighted** by **modified color factors**.

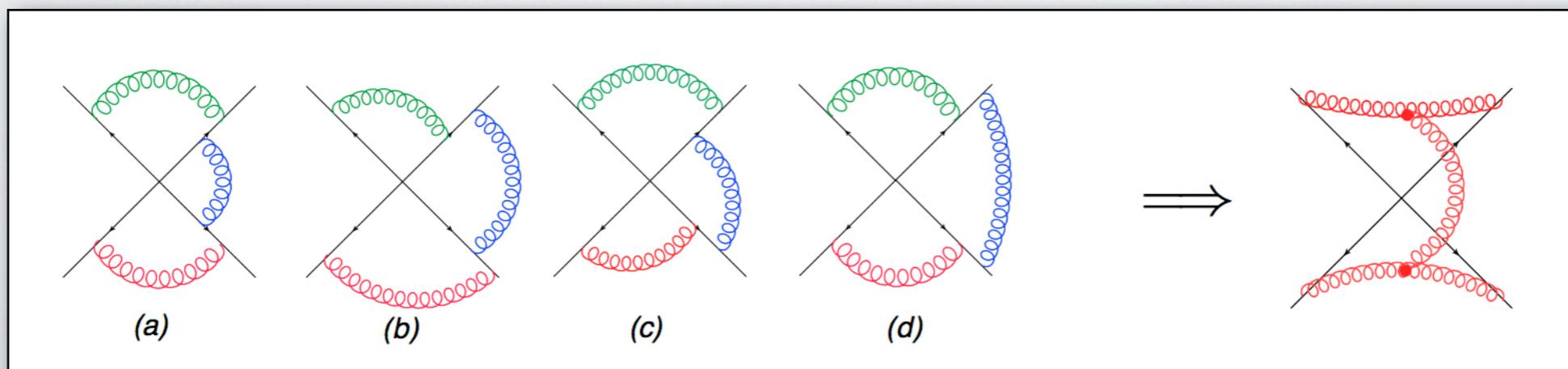
Writing each diagram as the product of its natural **color** factor and a **kinematic** factor

$$D = C(D)\mathcal{F}(D)$$

a **web** W can be expressed as a **sum of diagrams** in terms of a **web mixing matrix** R

$$W = \sum_D \tilde{C}(D)\mathcal{F}(D) = \sum_{D,D'} C(D')R(D',D)\mathcal{F}(D)$$

The **non-abelian exponentiation theorem** holds: each web has the color factor of a **fully connected** gluon subdiagram (Gardi, Smillie, White).



Beyond the eikonal

The **soft expansion** can be **organized beyond leading power** using either path integral techniques (Laenen, Stavenga, White 08) or diagrammatic techniques (Laenen, LM, Stavenga, White 10). The basic **idea** is **simple**, but the combinatorics **cumbersome**. For **spinors**

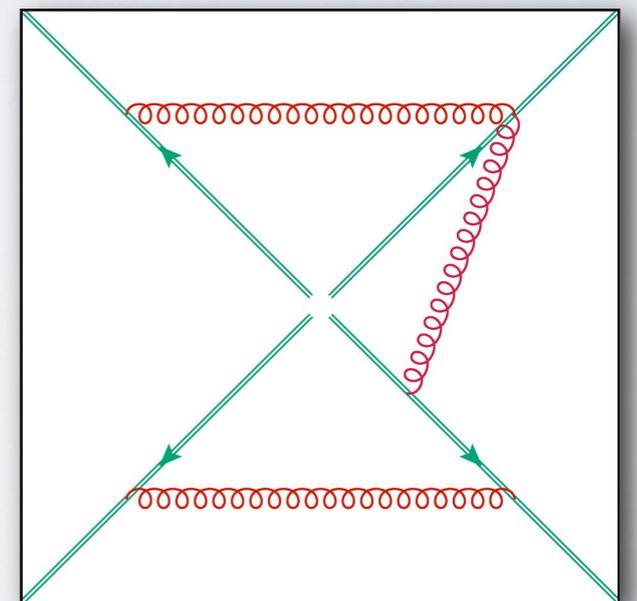
$$\frac{\not{p} + \not{k}}{2p \cdot k + k^2} \gamma^\mu u(p) = \left[\underbrace{\frac{p^\mu}{p \cdot k}}_{\text{Eikonal}} + \underbrace{\frac{\not{k} \gamma^\mu}{2p \cdot k}}_{\text{Spin-dependent}} - k^2 \underbrace{\frac{p^\mu}{2(p \cdot k)^2}}_{\text{Spin-independent}} \right] u(p) + \mathcal{O}(k)$$

- A class of **factorizable** contributions **exponentiate** via **NE webs**

$$\mathcal{M} = \mathcal{M}_0 \exp \left[\sum_{D_{\text{eik}}} \tilde{C}(D_{\text{eik}}) \mathcal{F}(D_{\text{eik}}) + \sum_{D_{\text{NE}}} \tilde{C}(D_{\text{NE}}) \mathcal{F}(D_{\text{NE}}) \right].$$

- **Feynman rules** exist for the **NE** exponent, including “**seagull**” vertices.

$$\mathcal{M} = \mathcal{M}_0 \exp [\mathcal{M}_{\text{eik}} + \mathcal{M}_{\text{NE}}] (1 + \mathcal{M}_r) + \mathcal{O}(\text{NNE}).$$



A next-to-eikonal web

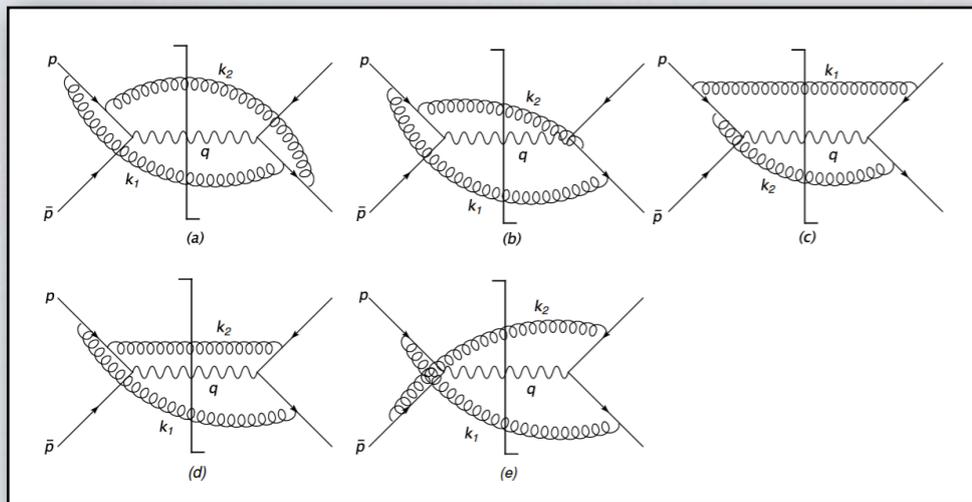
- **Non-factorizable** contributions involve **single gluon emission** from inside the **hard function**, and must be studied using **LBDK's theorem**.

Double real two-loop Drell-Yan

Multiple real emission contributions to EW annihilation processes involve **only factorizable** contributions. **NE Feynman rules** can be **tested** this level.

Defining the **Drell-Yan K-factor** as

$$K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz},$$



Real emission Feynman diagrams for the abelian part of the NNLO K-factor.

As a **test**, we (re)computed the C_F^2 part of **K** at **NNLO** from **ordinary Feynman diagrams**, and then **using NE Feynman rules**, finding complete **agreement**. As expected, plus distributions arise from the eikonal approximation.

Next-to-eikonal terms arise from **single-gluon** corrections: **seagull-type** contributions **vanish** for the **inclusive** cross section.

$$K_{\text{NE}}^{(2)}(z) = \left(\frac{\alpha_s}{4\pi} C_F\right)^2 \left[-\frac{32}{\epsilon^3} \mathcal{D}_0(z) + \frac{128}{\epsilon^2} \mathcal{D}_1(z) - \frac{128}{\epsilon^2} \log(1-z) \right. \\ \left. - \frac{256}{\epsilon} \mathcal{D}_2(z) + \frac{256}{\epsilon} \log^2(1-z) - \frac{320}{\epsilon} \log(1-z) \right. \\ \left. + \frac{1024}{3} \mathcal{D}_3(z) - \frac{1024}{3} \log^3(1-z) + 640 \log^2(1-z) \right],$$

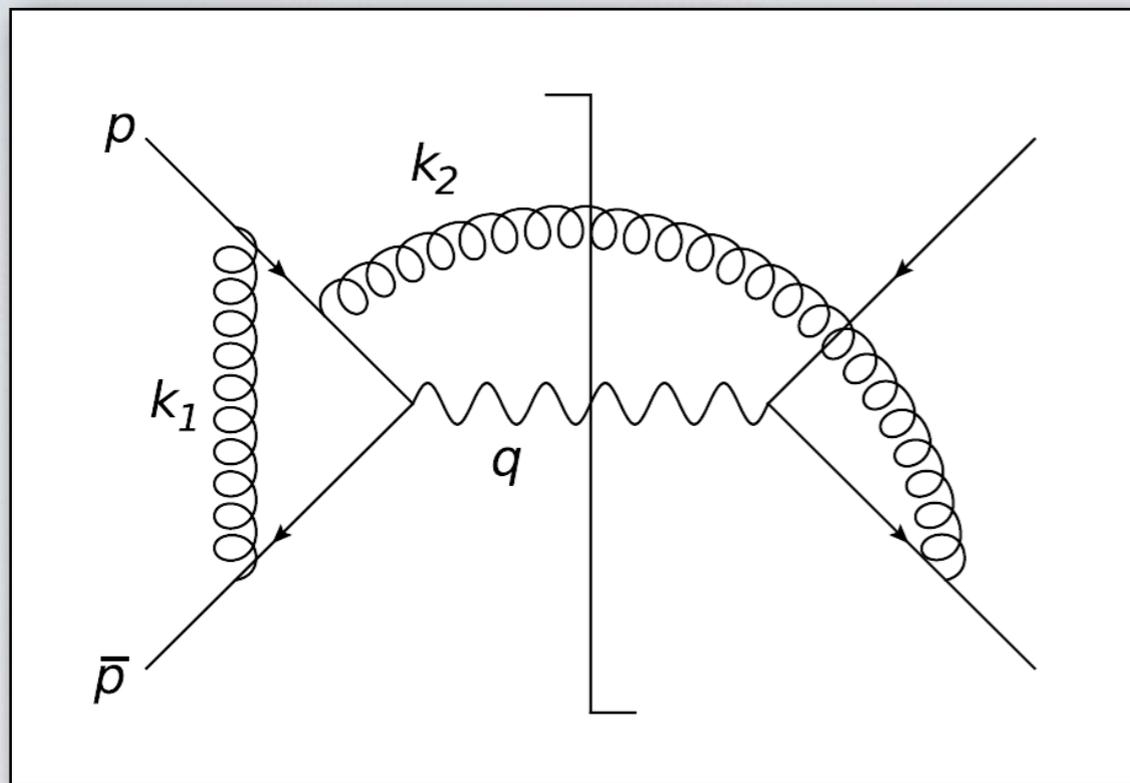
The abelian part of the NNLO K-factor from real emission, omitting constants

HARD COLLINEAR EMISSION



A collinear problem

Non-factorizable contributions start at **NNLO**. For **massive** particles they can be traced to the **original LBK** theorem. For **massless** particles a **new contribution** to **NLP** logs **emerges**.

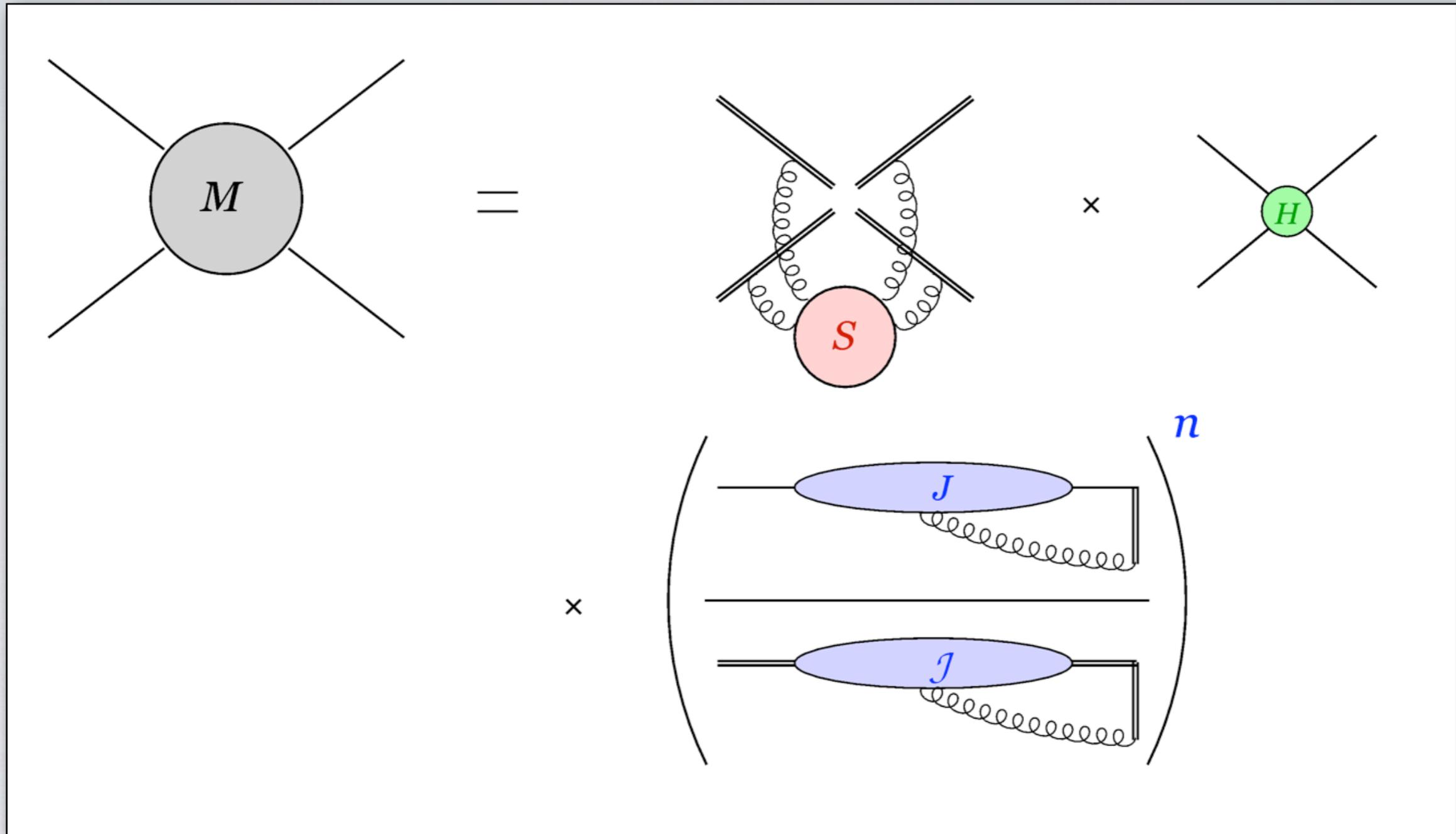


A Feynman diagram containing a collinear enhancement

- Gluon k_2 is **always** (next-to) **soft** for **EW** annihilation **near threshold**.
- When k_1 is (next-to) **soft** all logs are **captured** by **NE** rules.
- Contributions with k_1 **hard** and **collinear** are **missed** by the soft expansion.
- The **collinear pole** interferes with **soft emission** and generates **NLP** logs.
- The problem **first arises** at **NNLO**

- These contributions are **missed** by the **LBK** theorem: it applies to an **expansion** in E_k/m .
- They can be **analyzed** using the **method of regions**: the relevant **factor** is $(p \cdot k_2)^{-\epsilon}/\epsilon$.
- They **cause** the **breakdown** of **next-to-soft theorems** for amplitudes **beyond tree level**.
 ➡ the **soft** expansion and the limit $\epsilon \rightarrow 0$ **do not commute**.
- They **require** an **extension** of **LBK** to $m^2/Q < E_k < m$. It was **provided** by **Del Duca (90)**.

LP factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

Beyond Low's theorem

A **slightly modified** version of **Del Duca's** result gives the **radiative amplitude** in terms of the **non-radiative** one, its **derivatives**, and **two "jet"** functions.

$$\mathcal{A}^\mu(p_j, k) = \sum_{i=1}^2 \left\{ q_i \left(\frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} \right) + G_i^{\nu\mu} \left[\frac{J_\nu(p_i, k, n_i)}{J(p_i, n_i)} - q_i \frac{\partial}{\partial p_i^\nu} \left(\ln J(p_i, n_i) \right) \right] \right\} \mathcal{A}(p_i; p_j).$$

The tensors $G^{\mu\nu}$ **project out the eikonal** contribution present in the first term.

$$J(p, n, \alpha_s(\mu^2), \epsilon) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle ,$$

The **jet of ordinary IR** factorization.

$$J_\mu(p, n, k, \alpha_s(\mu^2), \epsilon) u(p) = \int d^d y e^{-i(p-k)\cdot y} \langle 0 | \Phi_n(y, \infty) \psi(y) j_\mu(0) | p \rangle ,$$

The **radiative jet**.

- At **tree level** the **radiative jet** displays the expected **dependence on spin**.
- **Dependence** on the **gauge vector** n^μ starts at **loop level**: **simplifications** arise for $n^2 = 0$.

$$\begin{aligned} J^{\nu(0)}(p, n, k) &= \frac{\not{k} \gamma^\nu}{2p \cdot k} - \frac{p^\nu}{p \cdot k} \\ &= -\frac{p^\nu}{p \cdot k} + \frac{k^\nu}{2p \cdot k} - \frac{i k_\alpha \Sigma^{\alpha\mu}}{2p \cdot k}. \end{aligned}$$

LOOKING TO THE FUTURE



A Perspective

- Perturbation theory continues to display **new** and **unexplored** structures.
- Leading power threshold resummation is **highly developed** and provides some of the **most precise** predictions in perturbative **QCD**.
- Mellin-space **constants** naturally **reside in the exponent** for simple processes.
- Low's theorem is the first of **many hints** that **NLP logs** can be understood and **organized**.
- Different approaches catch a number of **towers** of **NLP logs** in simple processes.
- The **next-to-soft** approximation is **well understood**, using both **diagrammatic** and **path integral** approaches, even for **multi-parton** processes.
- **Hard collinear** emissions **spoil** Low's theorem: a **new** radiative **jet function** emerges.
- A **complete treatment** of **NLP threshold logs** is at hand.
- **Much work to do** to organize a true **resummation** formula, even for **EW** annihilation: we have a more **intricate** “**factorization**”, we must make sure to control **double countings**.
- In order to achieve **complete generality**, we will need to include **final state jets**.

THANK YOU!