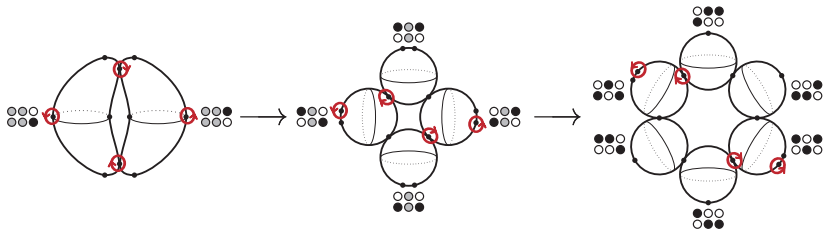


Two-loop Amplitudes from Maximal Unitarity

Kasper J. Larsen
ETH Zürich



Loopfest/RADCOR 2015
University of California, Los Angeles

June 17, 2015

(with Simon Caron-Huot, Henrik Johansson and David Kosower)

Two motivations for studying two-loop amplitudes:

- **Precision LHC phenomenology**

Quantitative estimates of QCD background: needed for precision measurements, uncertainty estimates of NLO calculations, and reducing renormalization scale dependence.

- **Geometric understanding of scattering amplitudes**

Fascinating connection to algebraic geometry and multivariate complex analysis.

Our aim is to extend generalized unitarity to two loops and express the two-loop amplitude in an integral basis directly.

Other approaches:

- Integrand reduction [Mastrolia, Mirabella, Ossola, Peraro], 2011 and [Badger, Frellesvig, Zhang], 2012 → **Giovanni Ossola's talk**
- Spinor integration techniques [Feng, Zhen, Huang, Zhou], 2014
- Iterated cuts [Abreu, Britto, Duhr, Gardi], 2014

The modern unitarity approach: basic unitarity (1/2)

Any one-loop amplitude can be decomposed into a basis of one-loop integrals

[Passarino, Veltman 1979]

$$A^{(1)} = c_1 \text{ (square) } + c_2 \text{ (triangle) } + c_3 \text{ (bubble) } + c_4 \text{ (self-energy) } + \text{rational terms}$$

thanks to integrand reductions, e.g. (using $\ell \cdot k_4 = \frac{1}{2} ((\ell + k_4)^2 - \ell^2)$)

$$[\ell \cdot k_4] = \frac{1}{2} \text{ (triangle) } - \frac{1}{2} \text{ (crossed triangle) }$$

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To determine $c_i \rightarrow$ apply (iterated) cuts to compute Disc in a specific channel.

[Bern, Dixon, Dunbar, Kosower 1995]

$$c_1 \text{ (box) } + c_2 \text{ (triangle) } + c_3 \text{ (bubble) } = \int d\text{LIPS} \text{ (cut diagrams) }$$

The modern unitarity approach: generalized unitarity (1/2)

The coefficients c_i in the basis decomposition

$$A^{(1)} = c_1 \text{ (square diagram)} + c_2 \text{ (triangle diagram)} + c_3 \text{ (figure-eight diagram)} + c_4 \text{ (circle diagram)} + \text{rational terms}$$

The equation shows the decomposition of the one-loop amplitude $A^{(1)}$ into four basis diagrams and rational terms. The diagrams are: a square with four external legs, a triangle with three external legs, a figure-eight (two circles joined at a point) with four external legs, and a circle with two external legs. Each diagram has dashed lines indicating external legs.

can be determined more efficiently by taking *generalized cuts*.

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Determine c_1 by applying quadruple cuts [Britto, Cachazo, Feng, 2004]:

$$\text{Four tree diagrams with red cuts} = c_1 \text{ (square diagram with red cuts)} \implies c_1 = \frac{1}{2} \sum_{\text{kin sols}} \prod_{j=1}^4 A_j^{\text{tree}}$$

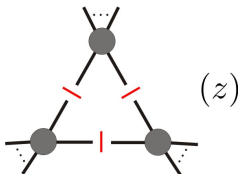
The modern unitarity approach: generalized unitarity (2/2)

A **triple cut** will leave $4 - 3 = 1$ **free complex parameter** z .

Parametrizing the loop momentum,

$$\ell^\mu = \alpha_1 K_1^{b\mu} + \alpha_2 K_2^{b\mu} + \frac{z}{2} \langle K_1^{b-} | \gamma^\mu | K_2^{b-} \rangle + \frac{\alpha_4(z)}{2} \langle K_2^{b-} | \gamma^\mu | K_1^{b-} \rangle$$

one obtains a formula for the triangle coefficient [Forde, 2007]

$$c_\Delta = \oint_{C(\infty)} \frac{dz}{z} \text{ (diagram)} (z)$$


From trees to two loops

Expand the massless 4-point two-loop amplitude in a basis, e.g.

$$A_4^{2\text{-loop}} = c_1(\epsilon) \left[\text{Diagram 1} \right] + c_2(\epsilon) \left[\text{Diagram 2} \right] + \text{ints with fewer props} + \text{rational terms}$$

The diagram on the left is a square with a vertical line through its center, and four external lines extending from the corners. The diagram on the right is identical but has two black dots on the horizontal lines, connected by a horizontal dashed line.

From trees to two loops

Expand the massless 4-point two-loop amplitude in a basis, e.g.

$$A_4^{2\text{-loop}} = c_1(\epsilon) \left[\text{Diagram 1} \right] + c_2(\epsilon) \left[\text{Diagram 2} \right] + \text{ints with fewer props} + \text{rational terms}$$

The diagram shows the expansion of the massless 4-point two-loop amplitude. The first term is $c_1(\epsilon)$ multiplied by a box diagram with two internal lines. The second term is $c_2(\epsilon)$ multiplied by a box diagram with two internal lines and two dots connected by a dashed line. To the right of the diagrams are the terms "+ ints with fewer props" and "+ rational terms".

Compute $c_1(\epsilon)$ and $c_2(\epsilon)$ according to



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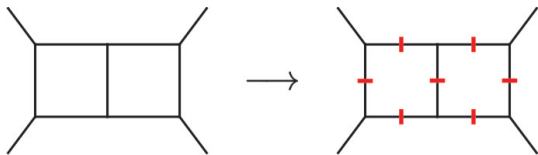


The machinery: *contour integrals* $\oint_{\Gamma_j}(\dots)$

The philosophy: basis integral $I_j \longleftrightarrow$ unique Γ_j producing c_j

The anatomy of two-loop maximal cuts

Cutting all seven visible propagators in the double-box integral,



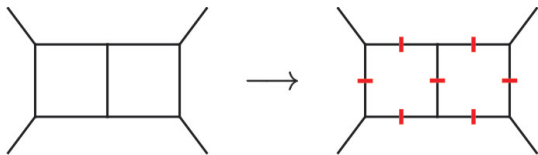
produces (cf. [Buchbinder, Cachazo] and [Kosower, KJL]), setting $\chi \equiv \frac{t}{s}$,

$$\int d^4 p d^4 q \prod_{i=1}^7 \frac{1}{\ell_i^2} \longrightarrow \int d^4 p d^4 q \prod_{i=1}^7 \delta^{\mathbb{C}}(\ell_i^2) = \oint_{\Gamma} \frac{dz}{z(z + \chi)},$$

a contour integral in the complex plane.

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a contour integral in the complex plane.

Jacobian poles $z = 0$ and $z = -\chi$: composite leading singularities

encircle $z = 0$ and $z = -\chi$ with $\Gamma = \omega_1 C_{\epsilon}(0) + \omega_2 C_{\epsilon}(-\chi)$

→ freeze z (“8th cut”)

Principle for selecting contours

To fix the contours, insist that

vanishing Feynman integrals must have vanishing generalized cuts.

This ensures that

$$I_1 = I_2 \quad \implies \quad \text{cut}(I_1) = \text{cut}(I_2).$$

Origin of terms with vanishing $\mathbb{R}^D \times \mathbb{R}^D$ integration:
reduction of Feynman diagram expansion to a *basis of integrals*
(including use of integration-by-parts identities [Chetyrkin and Tkachov],
1981).

Contour constraints, part 1/2

There are two classes of constraints on Γ 's:

1) Levi-Civita integrals. For example,

$$[\varepsilon(p, 1, 2, 4)] = 0 \implies [\varepsilon(p, 1, 2, 4)] = 0$$

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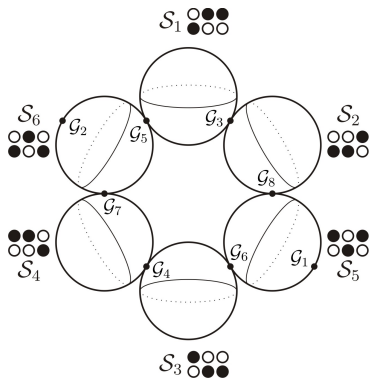
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2) integration-by-parts (IBP) identities must be preserved. For example,

$$\begin{aligned} & \text{Diagram} = \frac{\chi}{8} s_{12}^2 \text{Diagram} - \frac{3}{4} s_{12} \text{Diagram} + \dots \\ \implies & \text{Diagram} = \frac{\chi}{8} s_{12}^2 \text{Diagram} - \frac{3}{4} s_{12} \text{Diagram} \end{aligned}$$

The constraints in the case of four massless external momenta:



$$\omega_1 - \omega_2 = 0$$

$$\omega_3 - \omega_4 = 0$$

$$\omega_5 - \omega_6 = 0$$

$$\omega_7 - \omega_8 = 0$$

$$\omega_3 + \omega_4 - \omega_5 - \omega_6 = 0$$

$$\omega_1 + \omega_2 - \omega_5 - \omega_6 + \omega_7 + \omega_8 = 0$$

leaving $8 - 4 - 2 = 2$ free winding numbers.

Master contours: the concept

Going back to the two-loop basis expansion

$$A_4^{2\text{-loop}} = c_1(\epsilon) \left[\text{Diagram 1} \right] + c_2(\epsilon) \left[\text{Diagram 2} \right] + \text{ints with fewer props} + \text{rational terms}$$

The diagram on the left is a rectangle with two vertical internal lines and four external lines. The diagram on the right is the same rectangle with two dots on the right vertical line connected by a horizontal dotted line.

and applying a heptacut one finds

$$\left[\text{Diagram 1 with red ticks} \right] \left[\prod_{j=1}^6 A_j^{\text{tree}} \right] = c_1(\epsilon) \left[\text{Diagram 1 with red ticks} \right] + c_2(\epsilon) \left[\text{Diagram 2 with red ticks} \right]$$

The diagrams in this equation have red tick marks on all six internal lines. The diagram on the right also has a horizontal dotted line connecting two dots on the right vertical line.

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and applying a heptacut one finds

$$\left[\text{Diagram 1 with 7 red cuts} \right] \left[\prod_{j=1}^6 A_j^{\text{tree}} \right] = c_1(\epsilon) \left[\text{Diagram 1} \right] + c_2(\epsilon) \left[\text{Diagram 2} \right]$$

Exploit free parameters $\rightarrow \exists$ contours with

$$P_1 : (\text{cut}(I_1), \text{cut}(I_2)) = (1, 0)$$

$$P_2 : (\text{cut}(I_1), \text{cut}(I_2)) = (0, 1).$$

We call such P_i *master contours* (or projectors).

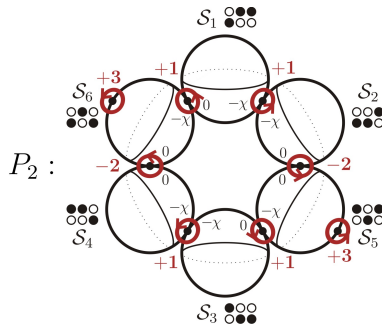
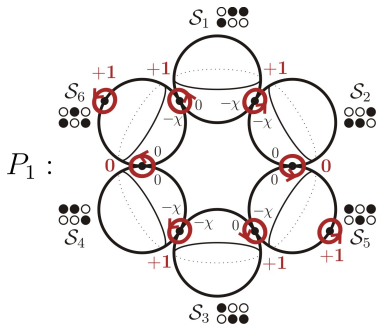
Master contours: results

With four massless external states,

$$c_1 = \frac{i\chi}{8} \oint_{P_1} \frac{dz}{z(z+\chi)} \prod_{j=1}^6 A_j^{\text{tree}}(z)$$

$$c_2 = -\frac{i}{4s_{12}} \oint_{P_2} \frac{dz}{z(z+\chi)} \prod_{j=1}^6 A_j^{\text{tree}}(z)$$

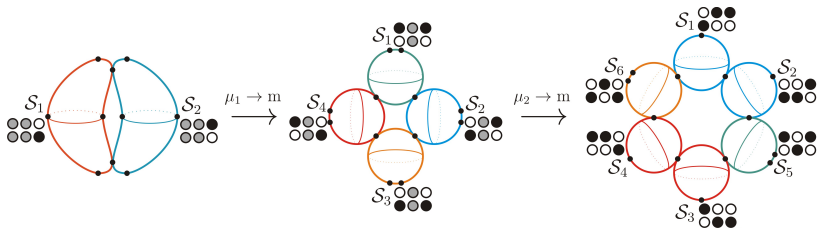
With our choice of basis integrals, the P_i are



$n =$ winding number

Double-box master contours at arbitrary multiplicity

Limits $\mu_i \rightarrow m \implies$ chiral branchings: torus $\xrightarrow{\mu_3 \rightarrow m}$

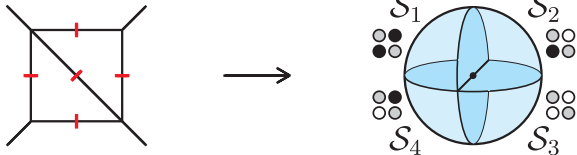


Each torus-pinching: new IR-pole + new residue thm
 \implies # of independent poles same in all cases

In all cases: **# of master Γ 's = # of basis integrals**
 \implies all linear relations are preserved
 \implies perfect analogy with one-loop generalized unitarity

Integrals with fewer propagators

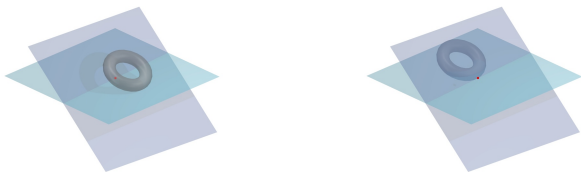
Solution to slashed-box on-shell constraints:



27 independent cycles (11 parity-conjugate pairs + 5 self-conjugate)

Multivariate residues depend on the contour of integration; e.g.,

$f(z_i) = \frac{z_1}{z_2(a_1 z_1 + a_2 z_2)(b_1 z_1 + b_2 z_2)}$ has two distinct cycles based at $(0, 0)$:



$\implies \exists$ maximal-cut contours that cannot be obtained from fewer cuts

\implies subtraction approach necessary

Maximal unitarity is a program aimed at automated computation of two-loop QCD amplitudes.

- One-to-one correspondence between two-loop master integrals and master contours
- Integration-by-parts identities “built into” contours
- $2 \rightarrow 2$ scattering needs $\mathcal{O}(100)$ residues to construct all integral coefficients; mild growth with multiplicity; produces compact expressions

Ongoing and future work:

- D -dimensional cuts
- Integrals with fewer propagators