Two-loop Amplitudes from Maximal Unitarity

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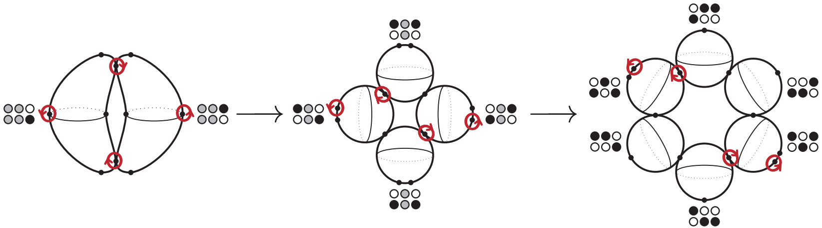
ETH Zürich



Loopfest/RADCOR 2015
University of California, Los Angeles

June 17, 2015

(with Simon Caron-Huot, Henrik Johansson and David Kosower)

Motivation and approaches

Two motivations for studying two-loop amplitudes:

- **Precision LHC phenomenology**
  Quantitative estimates of QCD background: needed for precision measurements, uncertainty estimates of NLO calculations, and reducing renormalization scale dependence.
- **Geometric understanding of scattering amplitudes**
  Fascinating connection to algebraic geometry and multivariate complex analysis.

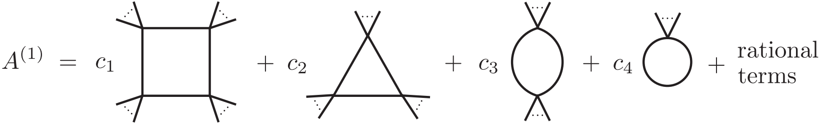
Our aim is to extend generalized unitarity to two loops and express the two-loop amplitude in an integral basis directly.

Other approaches:

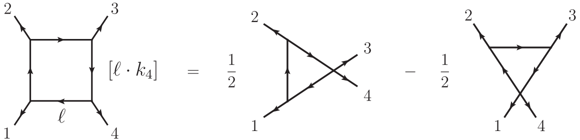
- Integrand reduction [Mastrolia, Mirabella, Ossola, Peraro], 2011 and [Badger, Frellesvig, Zhang], 2012 $\longrightarrow$ **Giovanni Ossola's talk**
- Spinor integration techniques [Feng, Zhen, Huang, Zhou], 2014
- Iterated cuts [Abreu, Britto, Duhr, Gardi], 2014

The modern unitarity approach: basic unitarity (1/2)

Any one-loop amplitude can be decomposed into a basis of one-loop integrals [Passarino, Veltman 1979]



thanks to integrand reductions, e.g. (using $\ell \cdot k_4 = \frac{1}{2}\left((\ell + k_4)^2 - \ell^2\right)$)



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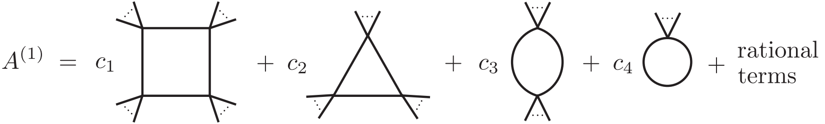

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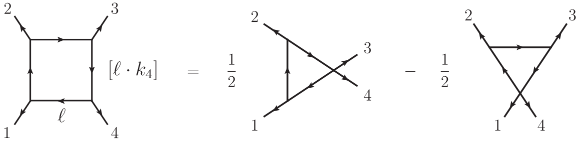
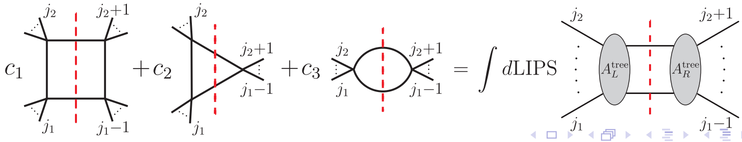
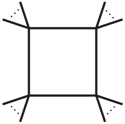
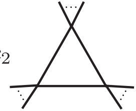
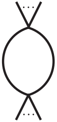


To determine $c_i \longrightarrow$ apply (iterated) cuts to compute Disc in a specific channel. [Bern, Dixon, Dunbar, Kosower 1995]



The coefficients c_i in the basis decomposition

$A^{(1)} = c_1$  $+\ c_2$  $+\ c_3$  $+\ c_4$  $+$ rational terms

can be determined more efficiently by taking *generalized cuts*.

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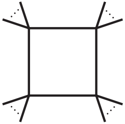
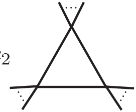
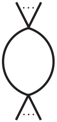
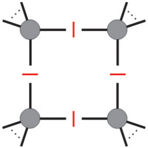
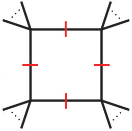
$A^{(1)} = c_1$  $+ c_2$  $+ c_3$  $+ c_4$  $+$ rational terms

can be determined more efficiently by taking *generalized cuts*.

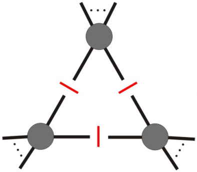
Determine c_1 by applying quadruple cuts [Britto, Cachazo, Feng, 2004]:

 $=$ c_1  $\Longrightarrow \quad c_1 = \frac{1}{2} \sum_{\text{kin sols}} \prod_{j=1}^{4} A_j^{\text{tree}}$

A triple cut will leave $4 - 3 = 1$ free *complex* parameter z.
Parametrizing the loop momentum,

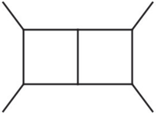
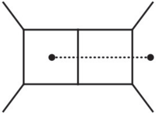
$$\ell^\mu = \alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \frac{z}{2}\langle K_1^{\flat-}|\gamma^\mu|K_2^{\flat-}\rangle + \frac{\alpha_4(z)}{2}\langle K_2^{\flat-}|\gamma^\mu|K_1^{\flat-}\rangle$$

one obtains a formula for the triangle coefficient [Forde, 2007]

$$c_\triangle = \oint_{C(\infty)} \frac{dz}{z}$$  $$(z)$$

From trees to two loops

Expand the massless 4-point two-loop amplitude in a basis, e.g.

$A_4^{2-\mathrm{loop}} \;=\; c_1(\epsilon)$  $+\; c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

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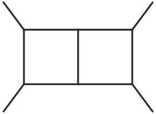
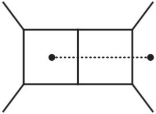
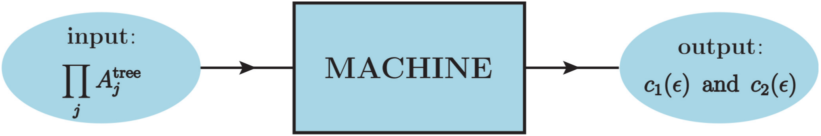
$A_4^{2-\text{loop}} = c_1(\epsilon)$  $+ \; c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

Compute $c_1(\epsilon)$ and $c_2(\epsilon)$ according to



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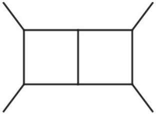
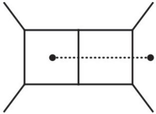
$A_4^{2-\text{loop}} = c_1(\epsilon)$  $+\, c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

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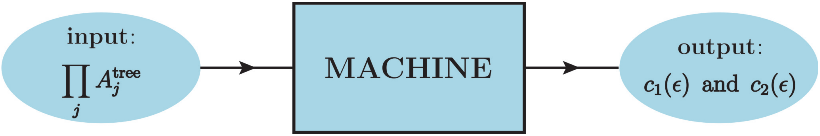
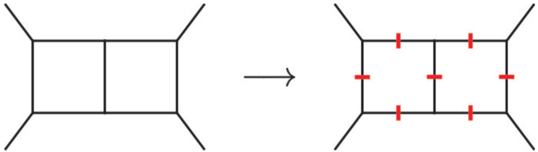


The machinery: *contour integrals* $\oint_{\Gamma_j}(\cdots)$

The philosophy: basis integral $I_j \longleftrightarrow$ unique Γ_j producing c_j

Cutting all seven visible propagators in the double-box integral,



produces (cf. [Buchbinder, Cachazo] and [Kosower, KJL]), setting $\chi \equiv \frac{t}{s}$,

$$\int d^4p\, d^4q \prod_{i=1}^{7} \frac{1}{\ell_i^2} \longrightarrow \int d^4p\, d^4q \prod_{i=1}^{7} \delta^{\mathbb{C}}(\ell_i^2) = \oint_{\Gamma} \frac{dz}{z(z+\chi)},$$

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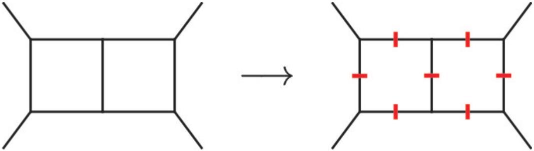

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Jacobian poles $z = 0$ and $z = -\chi$: composite leading singularities

encircle $z = 0$ and $z = -\chi$ with $\Gamma = \omega_1 C_\epsilon(0) + \omega_2 C_\epsilon(-\chi)$
$\longrightarrow$ freeze z ("8^{th} cut")

Principle for selecting contours

To fix the contours, insist that

vanishing Feynman integrals must have vanishing generalized cuts.

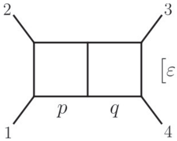
This ensures that

$$\mathrm{I}_1 = \mathrm{I}_2 \quad \Longrightarrow \quad \mathrm{cut}\,(\mathrm{I}_1) = \mathrm{cut}\,(\mathrm{I}_2)\,.$$

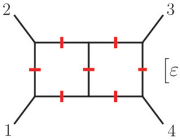
Origin of terms with vanishing $\mathbb{R}^D \times \mathbb{R}^D$ integration:
reduction of Feynman diagram expansion to a *basis of integrals*
(including use of integration-by-parts identities [Chetyrkin and Tkachov], 1981).

There are two classes of constraints on Γ's:

1) Levi-Civita integrals. For example,



$$[\varepsilon(p,1,2,4)] = 0 \quad \Longrightarrow$$



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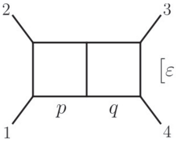

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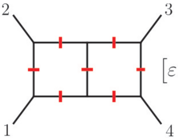

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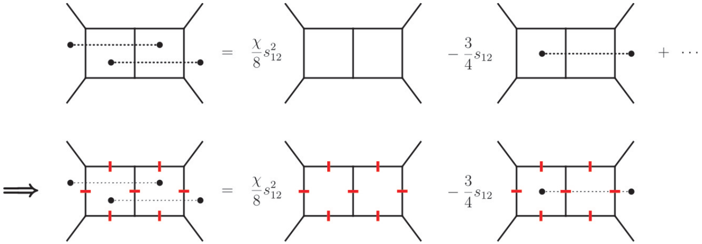
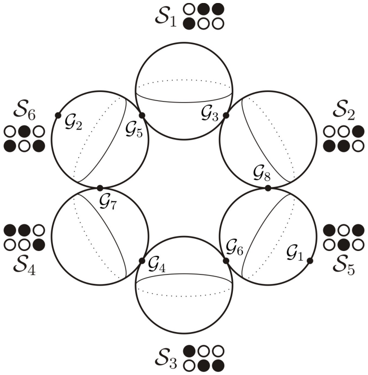
2) integration-by-parts (IBP) identities must be preserved. For example,



Contour constraints, part 2/2

The constraints in the case of four massless external momenta:



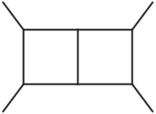
$$\omega_1 - \omega_2 = 0$$
$$\omega_3 - \omega_4 = 0$$
$$\omega_5 - \omega_6 = 0$$
$$\omega_7 - \omega_8 = 0$$
$$\omega_3 + \omega_4 - \omega_5 - \omega_6 = 0$$
$$\omega_1 + \omega_2 - \omega_5 - \omega_6 + \omega_7 + \omega_8 = 0$$

leaving $8 - 4 - 2 = 2$ free winding numbers.

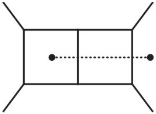
Master contours: the concept

Going back to the two-loop basis expansion

$A_4^{2-\text{loop}} = c_1(\epsilon)$



$+ \; c_2(\epsilon)$



$+$ ints with fewer props
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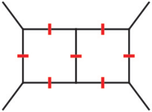
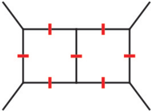
and applying a heptacut one finds



$\left[\prod_{j=1}^{6} A_j^{\text{tree}}\right] = c_1(\epsilon)$



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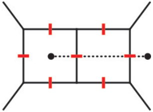

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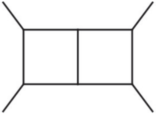
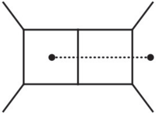
$A_4^{2-\text{loop}} = c_1(\epsilon)$  $+\, c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

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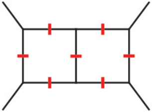
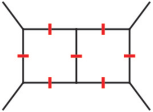
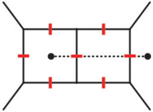
 $\left[\prod_{j=1}^{6} A_j^{\text{tree}}\right] = c_1(\epsilon)$  $+\, c_2(\epsilon)$ 

Exploit free parameters $\longrightarrow \exists$ contours with

$$P_1 : \ \big(\text{cut}(\text{I}_1),\, \text{cut}(\text{I}_2)\big) = (1,0)$$
$$P_2 : \ \big(\text{cut}(\text{I}_1),\, \text{cut}(\text{I}_2)\big) = (0,1)\,.$$

We call such P_i *master contours* (or projectors).

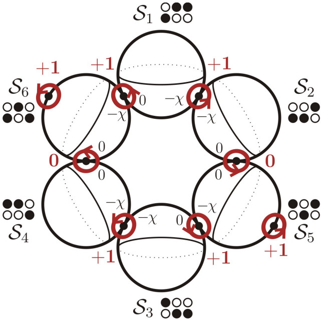
With four massless external states,

$$c_1 = \frac{i\chi}{8} \oint_{P_1} \frac{dz}{z(z+\chi)} \prod_{j=1}^{6} A_j^{\text{tree}}(z)$$

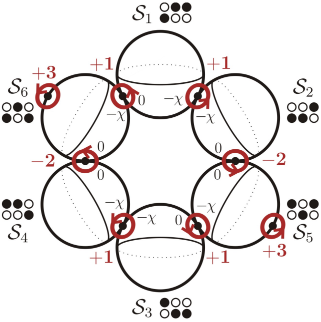
$$c_2 = -\frac{i}{4s_{12}} \oint_{P_2} \frac{dz}{z(z+\chi)} \prod_{j=1}^{6} A_j^{\text{tree}}(z)$$

With our choice of basis integrals, the P_i are

P_1 :



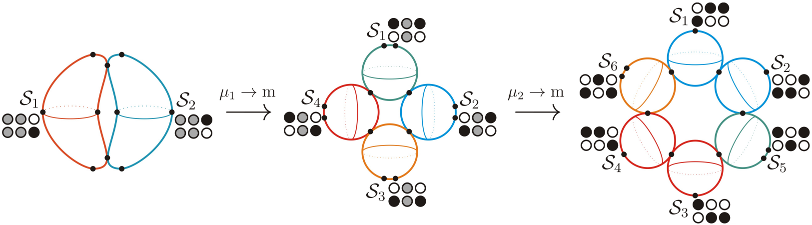
P_2 :



n = winding number

Double-box master contours at arbitrary multiplicity

Limits $\mu_i \to \mathrm{m} \implies$ chiral branchings: torus $\xrightarrow{\mu_3 \to \mathrm{m}}$



Each torus-pinching: new IR-pole + new residue thm

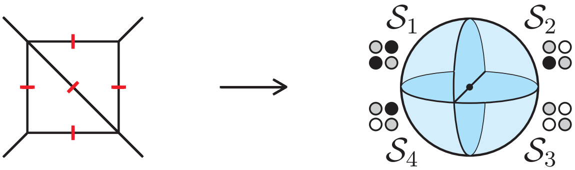
$\implies$ # of independent poles same in all cases

In all cases: **# of master Γ's = # of basis integrals**

$\implies$ all linear relations are preserved

$\implies$ perfect analogy with one-loop generalized unitarity

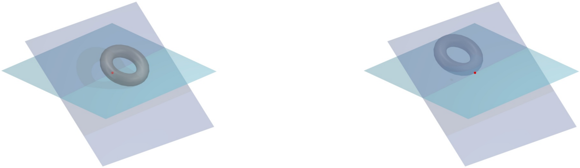
Solution to slashed-box on-shell constraints:



27 independent cycles (11 parity-conjugate pairs + 5 self-conjugate)

Multivariate residues depend on the contour of integration; e.g.,

$f(z_i) = \frac{z_1}{z_2(a_1 z_1 + a_2 z_2)(b_1 z_1 + b_2 z_2)}$ has two distinct cycles based at $(0,0)$:



$\Longrightarrow$ $\exists$ maximal-cut contours that cannot be obtained from fewer cuts

$\Longrightarrow$ subtraction approach necessary

Conclusions and outlook

Maximal unitarity is a program aimed at automated computation of two-loop QCD amplitudes.

- One-to-one correspondence between two-loop master integrals and master contours
- Integration-by-parts identities "built into" contours
- $2 \to 2$ scattering needs $\mathcal{O}(100)$ residues to construct all integral coefficients; mild growth with multiplicity; produces compact expressions

Ongoing and future work:

- D-dimensional cuts
- Integrals with fewer propagators