RADCOR-LOOPFEST 2015

LONG-DISTANCE SINGULARITIES IN MULTI-LEG SCATTERING AMPLITUDES

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NEW 3-LOOP RESULT FOR THE SOFT ANOMALOUS DIMENSION - WORK WITH ØYVIND ALMELID AND CLAUDE DUHR

19 JUNE, 2015

Long-distance singularities in multi-leg scattering amplitudes

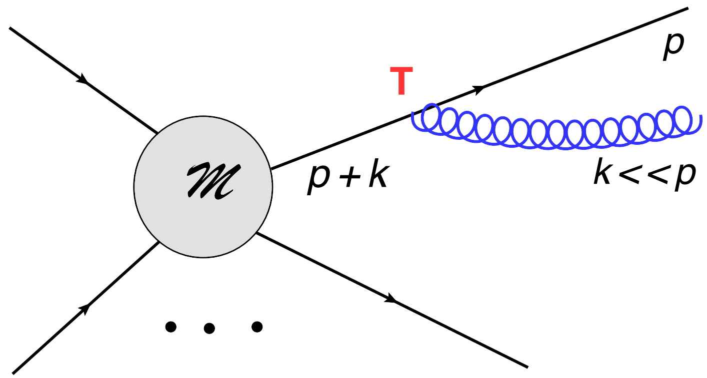
Plan of the talk

- Soft singularities: fixed-angle factorization, Wilson lines, rescaling symmetry.
- The soft anomalous dimension for massless partons: the dipole formula.
- Quadrupole interaction at 3-loop.
- Special kinematics: Regge limit, collinear limit.

The soft (eikonal) approximation and rescaling symmetry

Eikonal Feynman rules:

Assuming $k \ll p$ such that all components of k are small:



$$\bar{u}(p)\left(-\mathrm{i}g_s T^{(a)}\gamma^\mu\right)\frac{\mathrm{i}(\not{p}+\not{k}+m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon} \qquad \longrightarrow \qquad \bar{u}(p)g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}$$

Rescaling invariance: only the direction and the colour charge of the emitter matter.

$$g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon} = g_s T^{(a)}\frac{\beta^\mu}{\beta\cdot k+\mathrm{i}\varepsilon}$$

equivalent to emission from a Wilson line: $\Phi_{\beta_i}(\infty,0) \equiv P\exp\left\{\mathrm{i}g_s\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right\}$

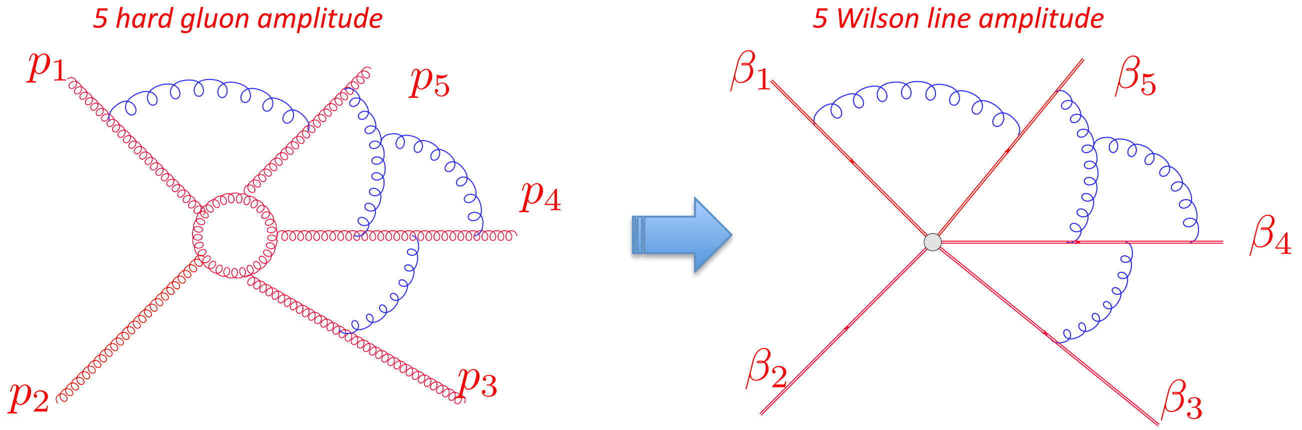
This symmetry is realised differently for lightlike and massive Wilson lines.

IR singularities from Wilson lines

Factorization at fixed angles:

all kinematic invariants are simultaneously taken large $p_i \cdot p_j = Q^2 \beta_i \cdot \beta_j \gg \Lambda^2$

Soft singularities factorise to all orders & computed from a product of Wilson lines:



$$\mathcal{M}_J(p_i, \epsilon_{\rm IR}) = \sum_K \mathcal{S}_{JK}\left(\gamma_{ij}, \epsilon_{\rm IR}\right) H_K(p_i)$$

$\mathcal{S}$ is a product of Wilson lines: $\mathcal{S} = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$

Due to rescaling symmetry it only depends on angles: $\gamma_{ij} = \dfrac{2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \, \beta_j^2}}$

IR singularities for amplitudes with massless legs

Exponentiation through solution of renormaliaztion-group equations:

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda^2/s_{ij},\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

The Dipole Formula:

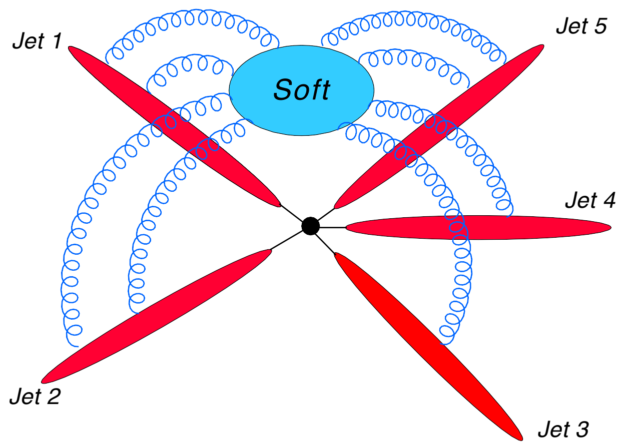
simple ansatz for the singularity structure of multi-leg massless amplitudes

$$\underset{\text{Dip.}}{\Gamma}(\lambda,\alpha_s) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathrm{T}_i\cdot\mathrm{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s\right)$$

Becher & Neubert,
EG & Magnea (2009)

Complete two-loop calculation by
Dixon, Mert-Aybat and Sterman in 2006
(confirming Catani's predictions from 1998).

Generalization to all orders motivated
by constraints based on **soft/jet factorisation**
and **rescaling symmetry.**



Factorization of amplitudes with massless legs

- Fixed angle scattering $p_i \cdot p_j = Q^2 \beta_i \cdot \beta_j \gg \Lambda^2$ with **lightlike partons** $p_i^2 = 0$

- IR singularities can be factorised - all originate in soft and collinear regions

$$\mathcal{M}_N\left(p_i/\mu, \epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \epsilon\right)\, H_L\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}\right)$$

Lightlike Wilson lines

$$\times \prod_{i=1}^{n} J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)$$

Jets (colour singlet)

- Double counting of soft-collinear region is removed by dividing by eikonal jets.

- Kinematic dependence of the soft function is now on $\beta_i \cdot \beta_j$, violating rescaling symmetry. This collinear anomaly is restored by the eikonal jets.
  This implies an **all-order constraint on the soft function,**
  leading to the **Dipole Formula.**

Becher & Neubert,
EG & Magnea (2009)

CORRECTIONS TO THE DIPOLE FORMULA

First possible corrections to the Dipole Formula:
Functions of conformally-invariant cross ratios at 3-loops, 4 legs:

$$\Gamma = \Gamma_{\mathrm{Dip.}} + \Delta(\rho_{ijkl})$$

$$\rho_{ijkl} = \frac{(p_i \cdot p_j)(p_k \cdot p_l)}{(p_i \cdot p_k)(p_j \cdot p_l)}$$

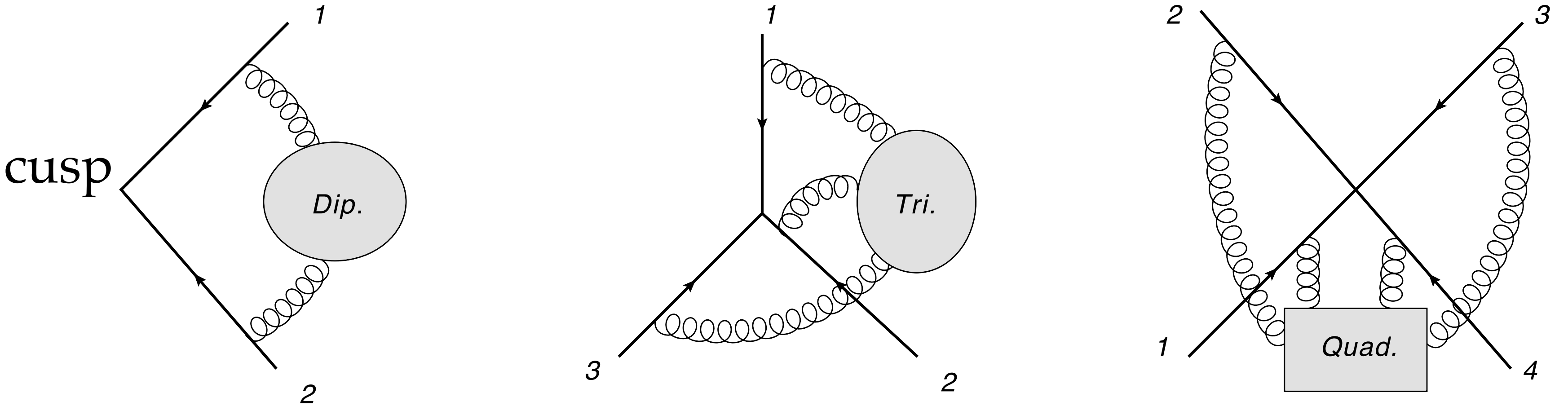
$\Delta(\rho_{ijkl})$ **is highly constrained by:**

- **Non-Abelian exponentiation**
- **Bose symmetry**
- **Transcendental weight**
- **Collinear limits**
- **Regge limit**

$$i f^{abe} i f^{cde} T_1^a T_2^b T_3^c T_4^d$$

EG & Magnea, Becher & Neubert (2009)
Dixon, EG & Magnea (2010)
Del Duca, Duhr, EG, Magnea & White (2011)
Ahrens & Neubert &Vernazza (2012)
Caron-Huot (2013)

The structure of the soft anomalous dimension: massless vs. massive partons



$T_1 \cdot T_2$ / planar $\qquad if^{abc} T_1^a T_2^b T_3^c \qquad if^{abe} if^{cde} T_1^a T_2^b T_3^c T_4^d$

massive	known @ 3-loop*	known @ 2-loops progress @ 3-loop	starts @ 3-loop - in progress
massless	known @ 3-loop	forbidden to all loops	**3-loop done!** **

* Grozin, Henn, Korchemsky & Marquard, Phys. Rev. Lett. 114, 062006 (2015)

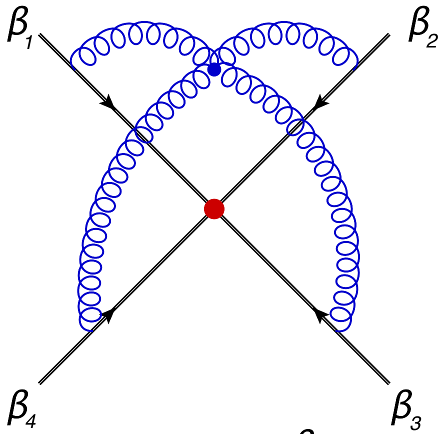
** **Almelid, Duhr, EG - to appear**

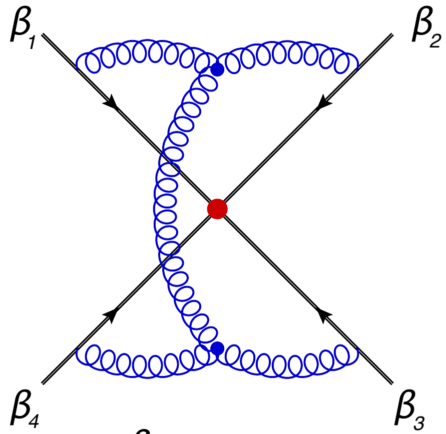
COMPUTING IR SINGULARITIES AT 3-LOOPS

Classes of three-loop webs connecting four Wilson lines

Single connected subgraph

Each web depends on all six angles - **can form conformally-invariant cross ratios** (cicrs)

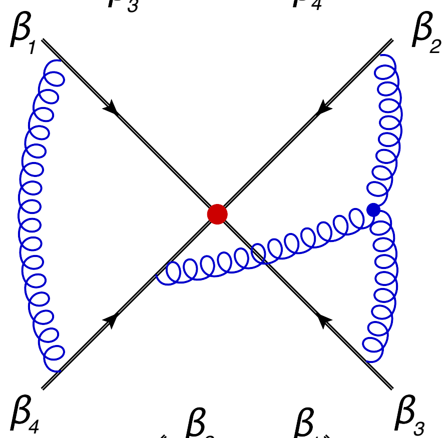




Two connected subgraphs

Depends on $\gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}$ only.

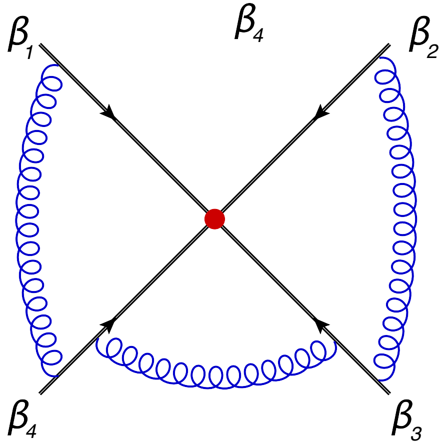
Cannot form cicrs - yields products of logs for near lightlike kinematics

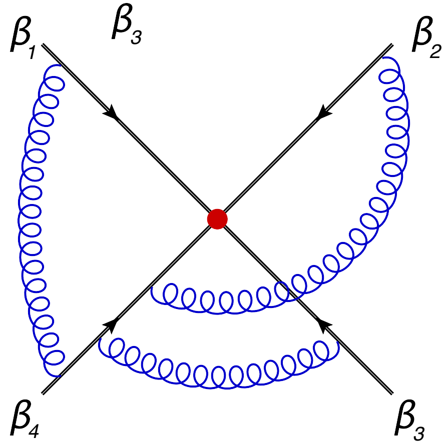


Three connected subgraphs (multiple gluon exchanges)

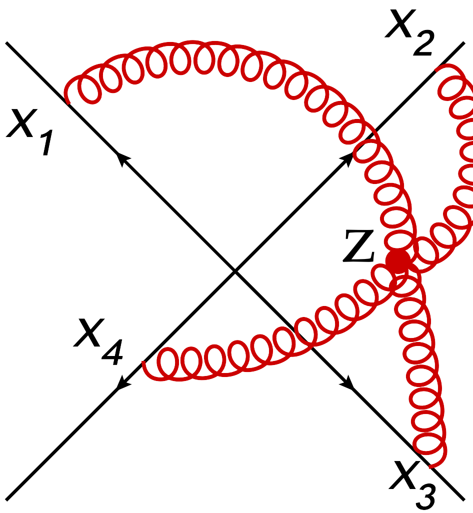
Depends on 3 angles only!

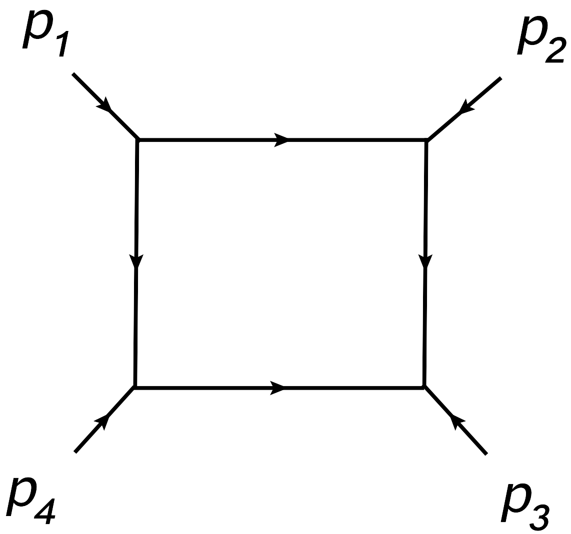
Cannot form cicrs - yields products of logs for near lightlike kinematics





Dual momentum box integral

$W_{4g} =$ 



Parametrise the positions along the Wilson lines by $x_i^\mu = \beta_i^\mu s_i$

Define auxiliary momenta $p_i = x_i - x_{i-1}$

The z integral is a 4-mass $\mathrm{Box}(p_1, p_2, p_3, p_4)$

$$C_{4g} = T_1^a T_2^b T_3^c T_4^d \left[f^{abe} f^{cde} (\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23}) + f^{ade} f^{bce} (\gamma_{12}\gamma_{34} - \gamma_{13}\gamma_{24}) + f^{ace} f^{bde} (\gamma_{12}\gamma_{34} - \gamma_{14}\gamma_{23}) \right]$$

$$W_{4g} = g_s^6 \mathcal{N}^4 C_{4g} \int_0^{\text{“}\infty\text{”}} ds_1 ds_2 ds_3 ds_4 \ \mathrm{Box}(x_1 - x_4, x_2 - x_1, x_3 - x_2, x_4 - x_3)$$

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \lambda \begin{pmatrix} ca \\ c(1-a) \\ (1-c)b \\ (1-c)(1-b) \end{pmatrix}$$

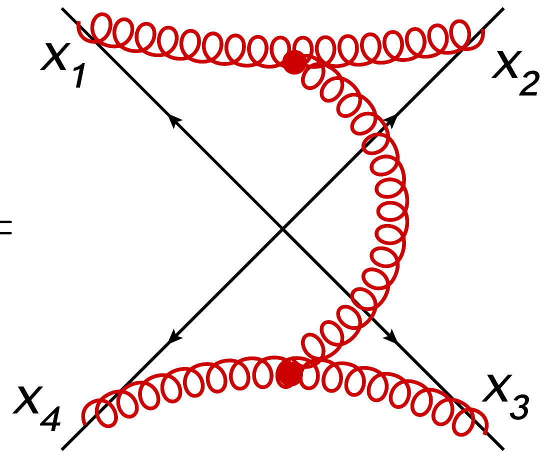
Integration over λ yields an overall $1/\epsilon$ UV pole.
Remaining integrations can be done in 4 dimensions.

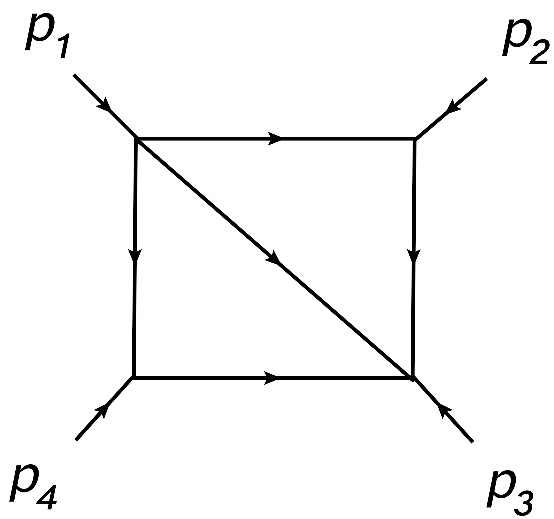
Ø. Almelid, C. Duhr, EG

Connected three-loop webs with two 3-gluon vertices

Ø. Almelid, C. Duhr, EG

A similar mapping - but with a diagonal box

$W_{(3g)^2} =$ 



W_{4g} and $W_{(3g)^2}$ may have non-trivial kinematic dependence in the limit $\beta_i^2 \to 0$

$$\rho_{ijkl} = \frac{\gamma_{ij}\,\gamma_{kl}}{\gamma_{ik}\,\gamma_{jl}} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$

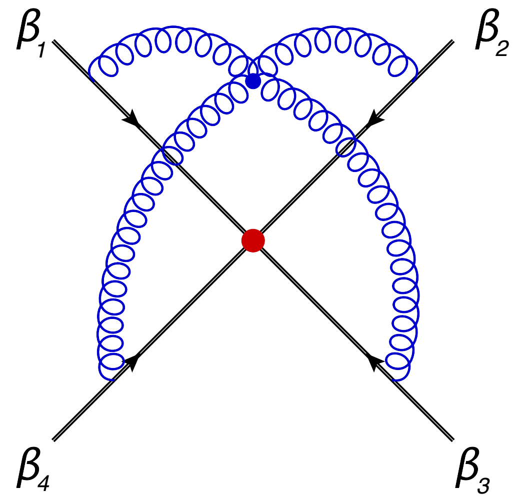
$$\rho_{1234} = z\bar{z}$$
$$\rho_{1432} = (1-z)(1-\bar{z})$$

We extract the asymptotic near-lightlike behaviour using the Mellin-Barnes technique. The remaining MB integral is three-fold, and can be converted into an iterated parameter integral and be expressed in terms of polylogarithms.

Connected webs: results and Bose symmetry

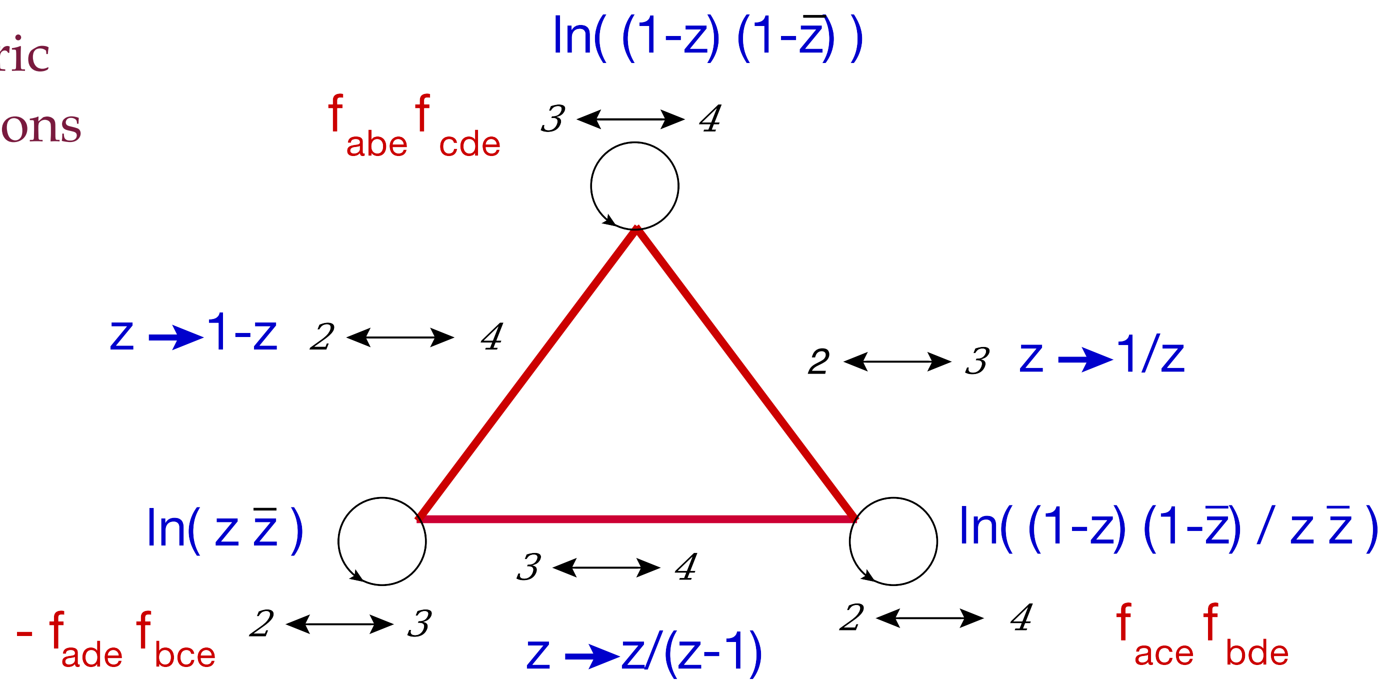
$$w_{4g}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\Big[f^{abe}f^{cde}\,(z\bar{z}-z-\bar{z})$$
$$+\, f^{ade}f^{bce}\,(1-z\bar{z}) + f^{ace}f^{bde}\,(1-z-\bar{z})\Big]\frac{1}{z-\bar{z}}\,g_1(z,\bar{z},\{\gamma_{ij}\})$$

$$\rho_{ijkl} = \frac{\gamma_{ij}\,\gamma_{kl}}{\gamma_{ik}\,\gamma_{jl}} = \frac{(\beta_i\cdot\beta_j)\,(\beta_k\cdot\beta_l)}{(\beta_i\cdot\beta_k)(\beta_j\cdot\beta_l)} \qquad \begin{aligned}\rho_{1234} &= z\bar{z}\\ \rho_{1432} &= (1-z)(1-\bar{z})\end{aligned}$$



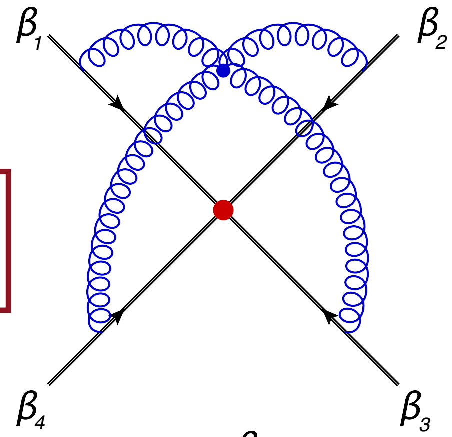
The permutation symmetry of the colour factors is mapped onto the kinematics

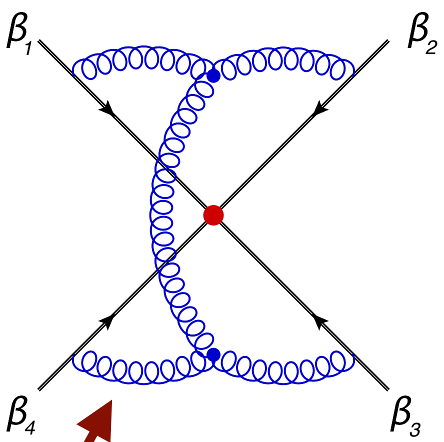
$g_1(z,\bar{z},\{\gamma_{ij}\})$ is symmetric under these transformations

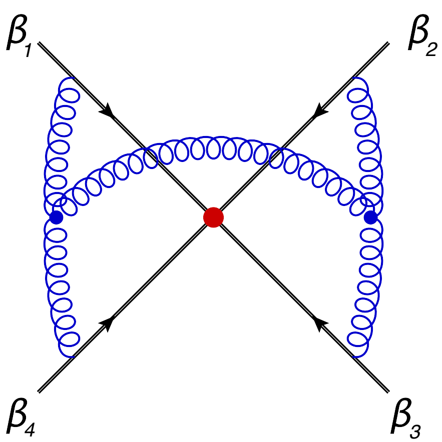


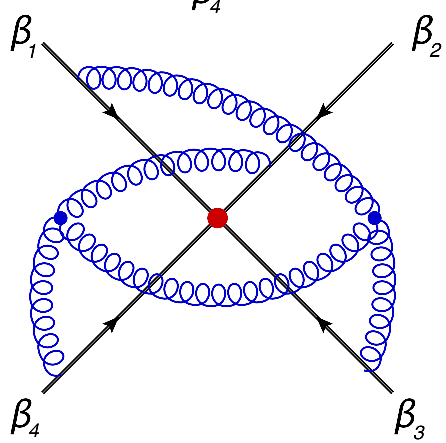
Summing the connected webs results

$$w_{4g}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f^{abe} f^{cde} \left(z\bar{z} - z - \bar{z}\right) + f^{ade} f^{bce} \left(1 - z\bar{z}\right) + f^{ace} f^{bde} \left(1 - z - \bar{z}\right) \Big] \boxed{\frac{1}{z-\bar{z}}\, g_1(z,\bar{z},\{\gamma_{ij}\})}$$









$$w_{(12)(34)}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \; f^{abe} f^{cde} \Big[g_0(z,\bar{z},\{\gamma_{ij}\}) - \boxed{\frac{z\bar{z}-z-\bar{z}}{z-\bar{z}}\, g_1(z,\bar{z},\{\gamma_{ij}\})} \Big]$$

cancels in the sum!

Pure function of uniform weight 5 ($\mathcal{N}$=4 SYM property)

Symbol alphabet $\{z, \bar{z}, 1-z, 1-\bar{z}\}$ relating to collinear/Regge limits

From the connected webs to the full quadrupole term in the soft anon. dim.

After applying Jacobi Identity one finds

$$w^{(3,-1)}_{\text{con.}} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \Big[f^{ade} f^{bce}\ \mathcal{F}_1{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) + f^{abe} f^{cde}\, \mathcal{F}_2{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) \Big]$$

and the functions <u>separate</u> into a polylogarithmic function of depending only on conformally invariant cross ratios via $\{z,\bar{z}\}$, and a function involving purely logarithmic dependence on individual cusp angles:

$$\mathcal{F}_n{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) = \mathcal{F}_n{}^{\text{con.}}(z,\bar{z}) + Q_n^{\text{con.}}(\{\log(\gamma_{ij})\})$$

<u>Rescaling symmetry</u> implies that the quadrupole contribution to the light-like soft anomalous dimension would depend <u>**exclusively**</u> on $\{z,\bar{z}\}$!

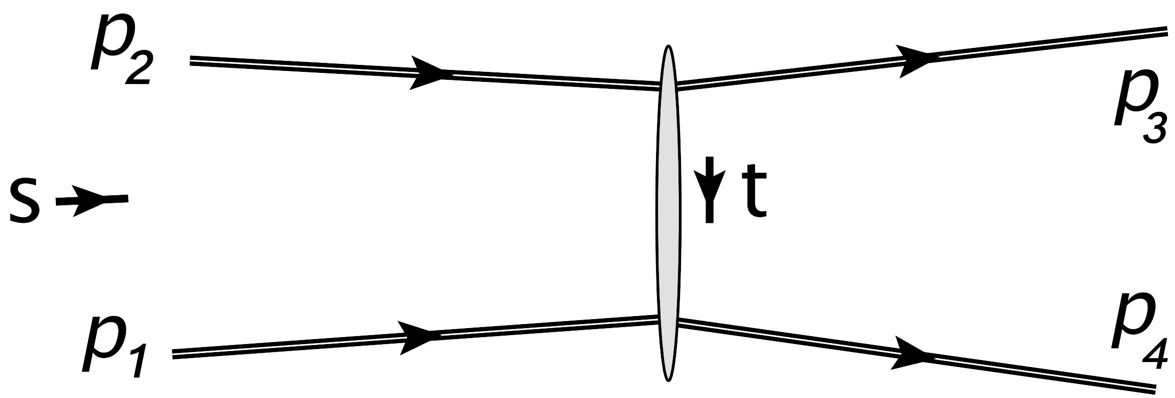
Indeed, we so far put aside all non-connected webs… These, in the light-like asymptotics, <u>**only involve logarithms**</u>, similarly to $Q_n^{\text{con.}}(\{\log(\gamma_{ij})\})$.
These must cancel any dependence on $\ln(\gamma_{ij})$ which isn't rescaling invariant.

One can infer the final, rescaling-invariant answer from connected webs alone!

Regge limit constraints

The high-energy limit is dominated by t-channel exchange.
Large Logs of (-t/s) are summed through Reggeization:

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$



Long-distance singularities of the Regge trajectory are controlled by

$$\alpha(t) = \frac{1}{4}(\mathbf{T}_2 + \mathbf{T}_3)^2 \int_0^{-t} \frac{d\lambda^2}{\lambda^2} \widehat{\gamma}_K(\alpha_s(\lambda^2, \epsilon))$$

Korchemskaya and Korchemsky (1996)
Del Duca, Duhr, EG, Magnea & White (2011)

Reggeization is proven through NLL, and the structure of the singularities (in the real part of the amplitude) is fully explained by the dipole formula. Therefore the **quadrupole term can start contributing at** $\mathrm{i}\alpha_s^3 \log^2(-t/s)$ or $\alpha_s^3 \log(-t/s)$

THE QUADRUPOLE FUNCTION

$$\Delta(z,\bar{z}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 \Big[f_{abe}f_{cde}\left(F\left(1-1/z\right) - F\left(1/z\right)\right) \\ - f_{ace}f_{bde}\left(F(z) - F(1-z)\right) \\ + f_{ade}f_{bce}\left(F\left(1/(1-z)\right) - F\left(1-1/(1-z)\right)\right)\Big]$$

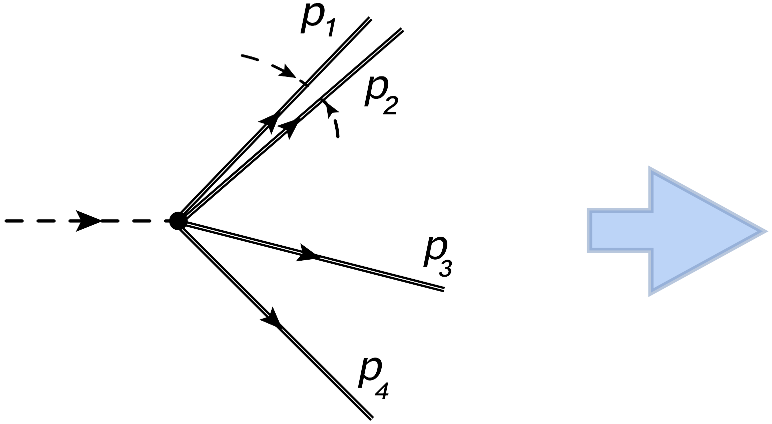
$$\boxed{F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2\Big(\mathcal{L}_{100}(z) + \mathcal{L}_{001}(z)\Big) + 6\zeta_4\,\mathcal{L}_1(z)}$$

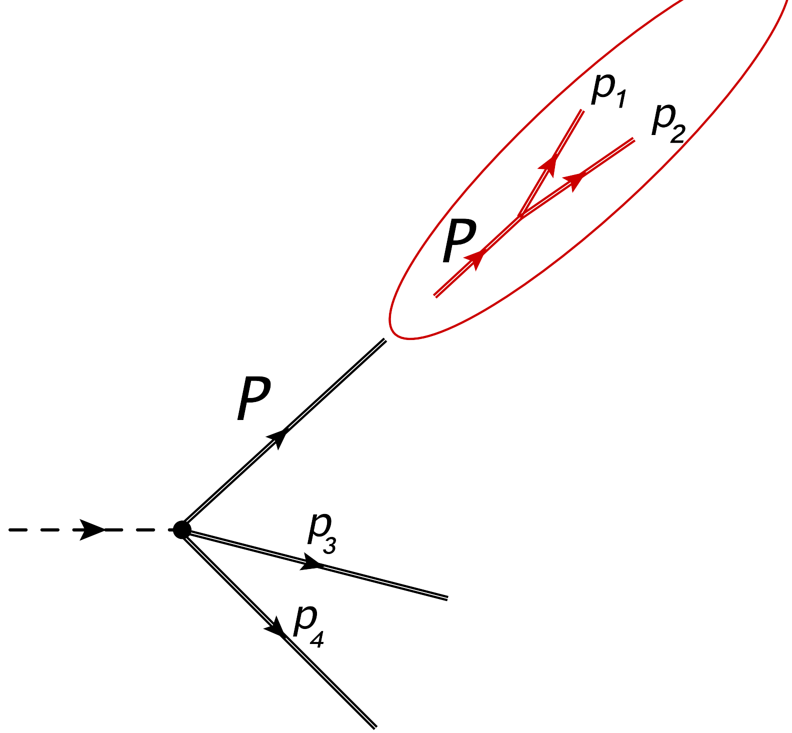
Ø. Almelid, C. Duhr, EG

$\mathcal{L}_{10\ldots}(z)$ are the single-valued harmonic polylogarithms introduced by Francis Brown in 2009. They are defined in the region where $\bar{z} = z^*$

THE COLLINEAR LIMIT

$$\mathcal{M}_L\left(p_1, p_2, \{p_j\}; \mu\right) \xrightarrow{1\|2} \mathbf{Sp}\left(p_1, p_2; \mu\right) \mathcal{M}_{L-1}\left(P, \{p_j\}; \mu\right)$$





In particular, IR singularities of the splitting amplitude are those present in L parton scattering (with 1| |2) and not in L-1 parton scattering:

$$\Gamma_{\mathbf{Sp}} = \Gamma_L - \Gamma_{L-1}$$

Surprise in the collinear limit

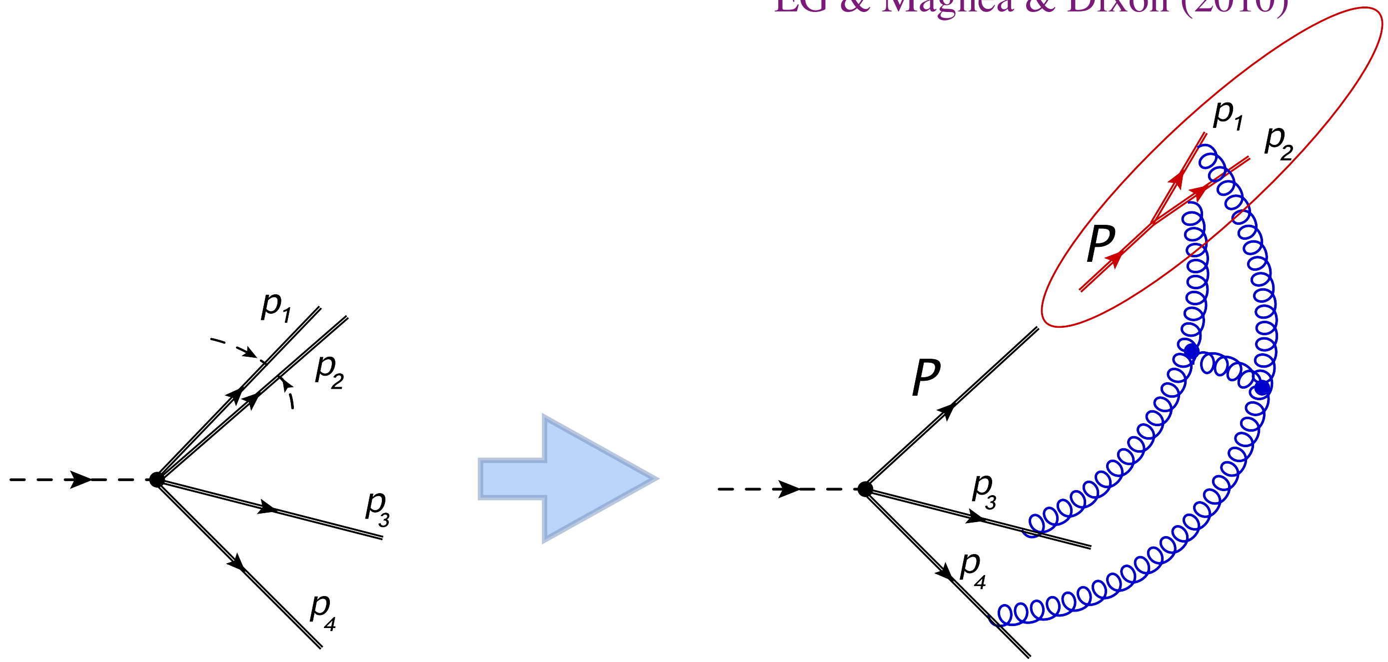
Previous expectation: the splitting amplitude should not depend on the rest of the process, thus $\Delta(z,\bar{z})$ should vanish in any collinear limit.

Becher & Neubert (2009)
EG & Magnea & Dixon (2010)

We find, however, for

$p_1 \to xP$

$p_2 \to (1-x)P$



$$\Delta(z,\bar{z}) \overset{1\|2}{\longrightarrow} -\left(\frac{\alpha_s}{\pi}\right)^3 \sum_{j,k\neq\{1,2\}} \left(-3\zeta_4 \ln\left(\frac{P^2}{2x(1-x)}\frac{p_j\cdot p_k}{(P\cdot p_j)(P\cdot p_k)}\right)+2\zeta_2\zeta_3+\zeta_5\right)\;[[T_1\cdot T_2, T_1\cdot T_j], T_2\cdot T_k]$$

At 3-loops (beyond the planar limit — *c.f.* Kosower 1999) the splitting amplitude **resolves the colour and directions of the rest of the process**!

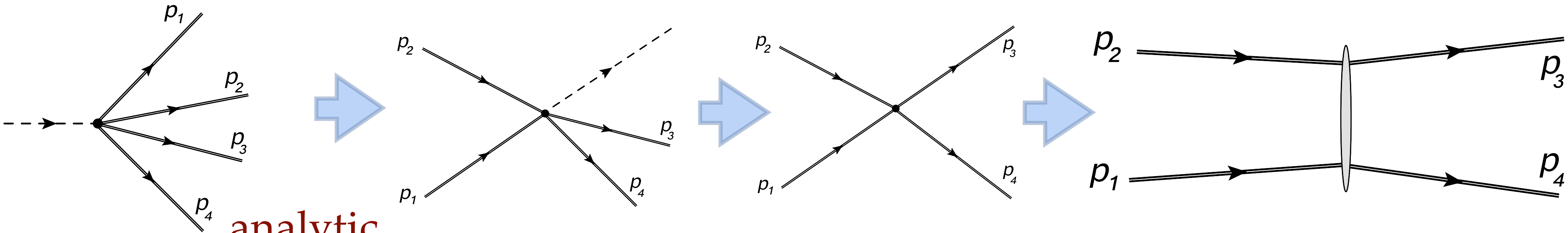
CONCLUSIONS

- IR singularities of massless scattering amplitudes are now known to **3-loops**.
- The first correction to the dipole formula takes the form of a quadrupole interaction simultaneously correlating colour and kinematics of 4 patrons.
- The quadrupole term is expressed in terms of single-valued harmonic polylogarithms of weight 5, depending on $\{z, \bar{z}\}$. These variables are simple algebraic functions of conformally-invariant cross ratios, and they manifest the symmetries and analytic structure of the quadruple interaction.
- Contrary to previous expectations, 3-loop splitting amplitudes acquire sensitivity to the colour flow and directions of the remaining partons.

Many thanks to my students and collaborators!

Analytic structure explored by taking the Regge limit

Euclidean | Scattering | momentum conserving | Regge limit

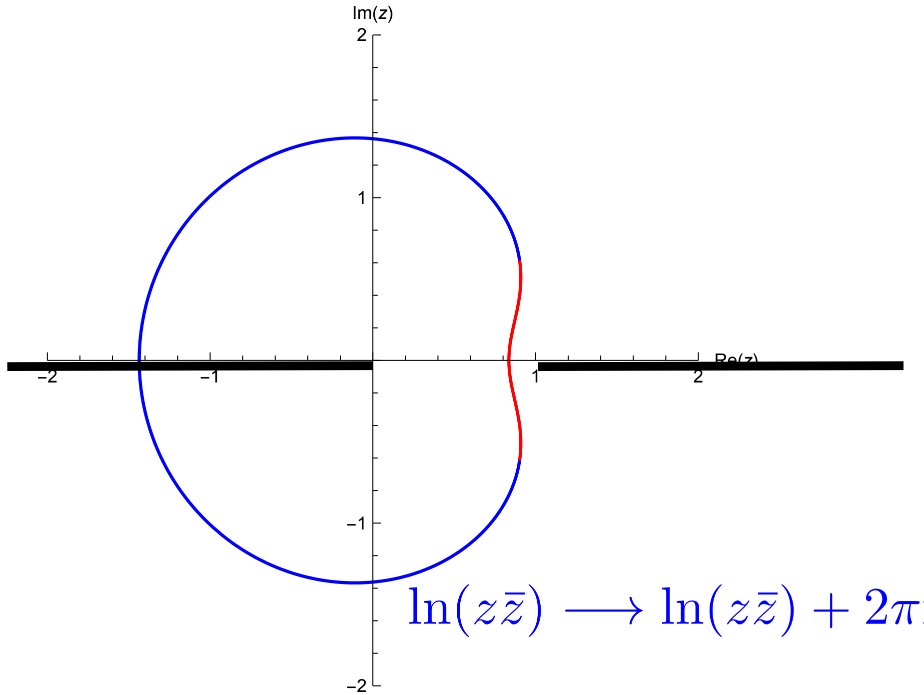


analytic continuation

$$z = \bar{z} = s/(s+t)$$

$$z = \bar{z} \to 1$$

$$-s_{12} = -s_{34} = s\mathrm{e}^{\mathrm{i}\pi}$$
$$-s_{23} = -s_{14} = -t > 0$$
$$-s_{13} = -s_{14} = -u = s + t > 0$$



$$z\bar{z} = \frac{(-s_{12})(-s_{34})}{(-s_{13})(-s_{24})} = \frac{s^2\mathrm{e}^{\mathrm{i}2\pi}}{(s+t)^2}$$

$$(1-z)(1-\bar{z}) = \frac{(-s_{14})(-s_{23})}{(-s_{13})(-s_{24})} = \frac{(-t)^2}{(s+t)^2}$$

Regge limits in different channels correspond to discontinuities around $z = 0, 1, \infty$

Multiple gluon exchange webs

The combinations of diagrams appearing in the exponent (subtracted webs) are **much simpler** than individual diagrams:

$$\Gamma^{(n)} \ni \overline{w}^{(n,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^n C_{i_1,i_2,\ldots i_{n+1}} \int dx_1 dx_2 \ldots dx_n \times \prod_{k=1}^{n} p_0(x_k,\alpha_k) \times$$
$$\mathcal{G}_{n-1}\Big(x_1, x_2, \ldots, x_n; q(x_1,\alpha_1), q(x_2,\alpha_2), \ldots q(x_n,\alpha_n)\Big)$$

$$p_0(x,\alpha_{ij}) = \frac{2\hat{\beta}_i\cdot\hat{\beta}_j}{(x\hat{\beta}_i-(1-x)\hat{\beta}_j)^2} = r(\alpha_{ij})\left[\frac{1}{x-\frac{1}{1-\alpha_{ij}}} - \frac{1}{x+\frac{\alpha_{ij}}{1-\alpha_{ij}}}\right]$$

Conjecture:

(1) $\mathcal{G}_{n-1}$ is made exclusively of powers of logs and Heaviside functions.

(2) $\overline{w}^{(n,-1)}$ is a sum of products of polylogs each depending on a single α each having a Symbol with alphabet $\{\alpha, 1-\alpha^2\}$

EG (arXiv:1310.5268)
JHEP 1404 (2014) 044

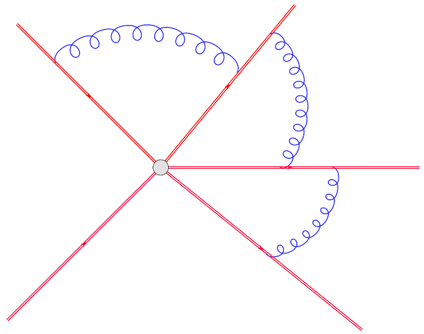
A restricted basis of harmonic polylogarithmic functions was constructed, which is conjectured to be sufficient for **all** multiple gluon exchange webs!

Falcioni, EG, Harley, Magnea, White
(arXiv:1407.3477) JHEP 1410 (2014) 10

Computing IR singularities using non-lightlike Wilson lines

- Product of non-lightlike Wilson lines is multiplicatively renormalizable

$$S = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$$



Arefeva, Dotsenko-Vergeles, Brandt-Neri-Sato (81)

- IR - UV relation:

$$\mathcal{S}\left(\gamma_{ij}, \epsilon_{\mathrm{IR}}\right) = \underbrace{\mathcal{S}_{\mathrm{UV+IR}}}_{=1} Z\left(\gamma_{ij}, \epsilon_{\mathrm{UV}}\right)$$

Korchemsky-Radyushkin (87)

- To determine the renormalization Z of a product of Wilson lines we put **an exponential IR cutoff** along the Wilson lines and compute the UV singularity in dimensional regularisation, $D = 4 - 2\epsilon, \quad \epsilon > 0$
- Define the soft anomalous dimension $\dfrac{dZ}{d\ln\mu} = -Z\Gamma$
- The anomalous dimension is determined by the coefficient of the single $1/\epsilon$ UV pole of Z - it is independent of the IR regularisation!

Diagrammatic soft gluon exponentiation

Webs are the diagrams contributing to w, the exponent of the corrector

$$\mathcal{S} = \exp w = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$$

- Abelian case (1961)
  Only connected diagrams contribute to the exponent

  Yennie-Frautchi-Suura

- Non-abelian, colour singlet (two-line) case (1983)
  Only irreducible diagrams contribute to the exponent

  Gatheral
  Frenkel & Taylor

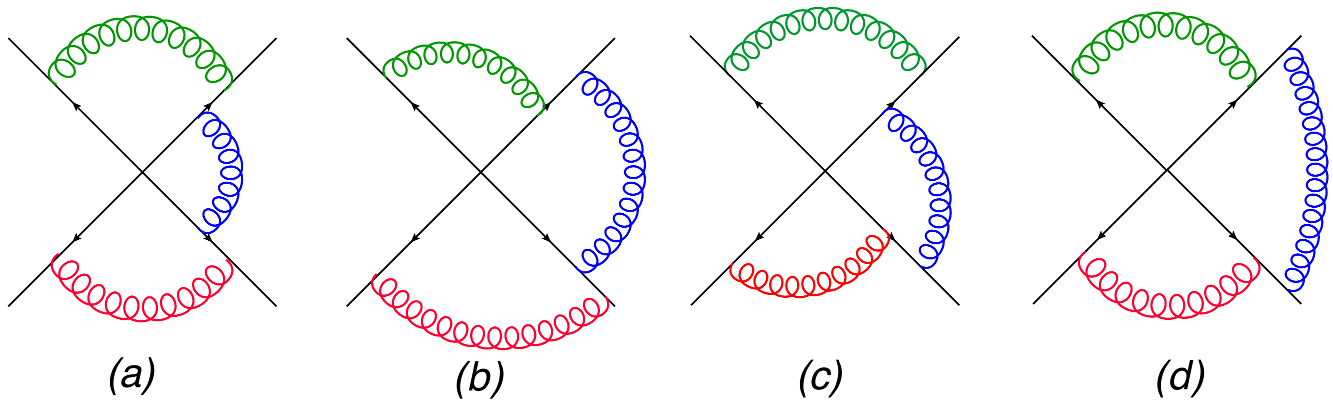
- Non-abelian, multi-line case (2010)
  Also reducible diagrams contribute -
  webs are formed by sets of diagrams.

  EG, Leanen, Stavenga & White
  Mitov, Sterman & Sung

<u>Theorem</u>: all colour structures in the exponent correspond to connected graphs

EG, Smillie & White (2013)

Three-loop web: example

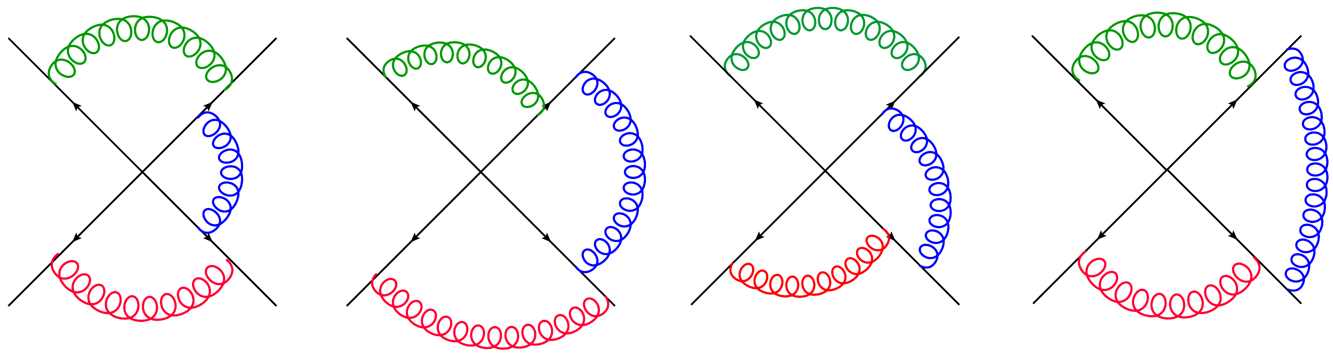


The entire web contributes:

$$= \begin{pmatrix} \mathcal{F}(3a) \\ \mathcal{F}(3b) \\ \mathcal{F}(3c) \\ \mathcal{F}(3d) \end{pmatrix}^T \frac{1}{6} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} C(3a) \\ C(3b) \\ C(3c) \\ C(3d) \end{pmatrix}$$

Kinematics Web mixing matrix R Colour

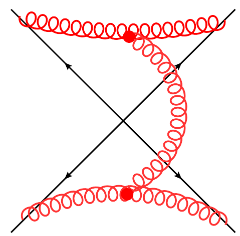
Three-loop web: example



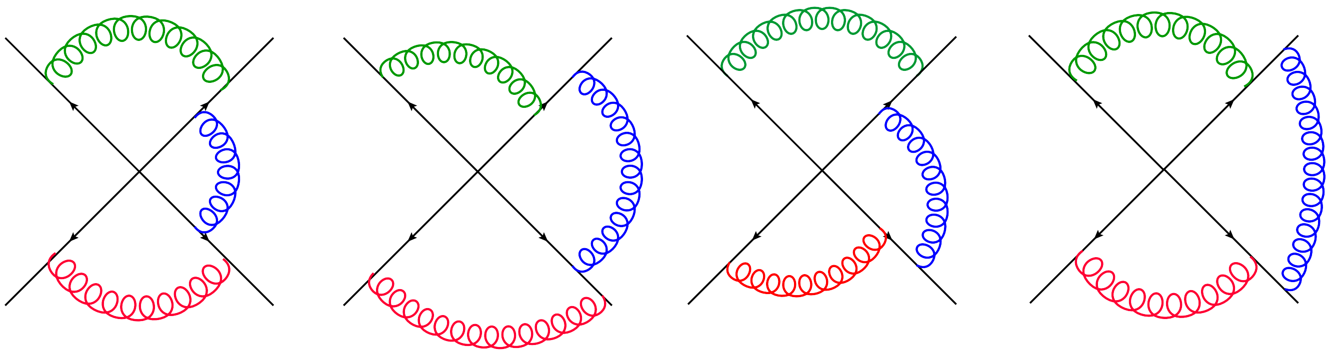
(a) (b) (c) (d)

$$
= \begin{pmatrix} \mathcal{F}(3a) \\ \mathcal{F}(3b) \\ \mathcal{F}(3c) \\ \mathcal{F}(3d) \end{pmatrix}^T \frac{1}{6} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} C(3a) \\ C(3b) \\ C(3c) \\ C(3d) \end{pmatrix}
$$

$$
= \underbrace{\frac{1}{6}\Big(\mathcal{F}(3a) - 2\mathcal{F}(3b) - 2\mathcal{F}(3c) + \mathcal{F}(3d)\Big)}_{\text{subdivergences cancel}} \times \underbrace{\Big(C(3a) - C(3b) - C(3c) + C(3d)\Big)}
$$



THREE-LOOP WEB RESULT



Very simple structure: sum of products of polylogs of individual angles!

$$\overline{w}^{(3)}_{1221} = -\, f^{abe} f^{cde} T_1^a T_2^b T_3^c T_4^d \, \left(\frac{\alpha_s}{4\pi}\right)^3 r(\alpha_{12}) r(\alpha_{23}) r(\alpha_{34})$$

$$\alpha_{ij} + \frac{1}{\alpha_{ij}} = \frac{-2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2\, \beta_j^2}} = -\gamma_{ij}$$

$$\Bigg[-8U_2(\alpha_{12}) \ln\alpha_{23} \ln\alpha_{34} - 8U_2(\alpha_{34}) \ln\alpha_{12} \ln\alpha_{23} + 16\Big(U_2(\alpha_{23}) - 2\Sigma_2(\alpha_{23})\Big) \ln\alpha_{12} \ln\alpha_{34}$$

$$-2\ln\alpha_{12}\, U_1(\alpha_{23})\, U_1(\alpha_{34}) - 2\ln\alpha_{34}\, U_1(\alpha_{12})\, U_1(\alpha_{23}) + 4\ln\alpha_{23}\, U_1(\alpha_{12})\, U_1(\alpha_{34}) \Bigg]$$

EG (arXiv:1310.5268)
JHEP 1404 (2014) 044

$$\mathcal{S}\left[U_1(\alpha)\right] = -4\alpha \otimes \frac{\alpha}{1-\alpha^2}$$

$$\mathcal{S}\left[U_2(\alpha)\right] = 4\alpha \otimes \frac{\alpha}{1-\alpha^2} \otimes \frac{\alpha}{1-\alpha^2}$$

$$\mathcal{S}\left[\Sigma_2(\alpha)\right] = 2\alpha \otimes \alpha \otimes \alpha$$

special points:

$\alpha = 0$	lightlike limit
$\alpha = -1$	production at threshold
$\alpha = 1$	straight line