Latest results on the 3-loop heavy flavor Wilson coefficients in DIS

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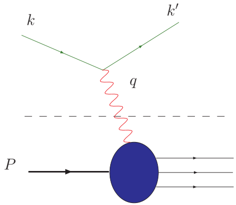
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

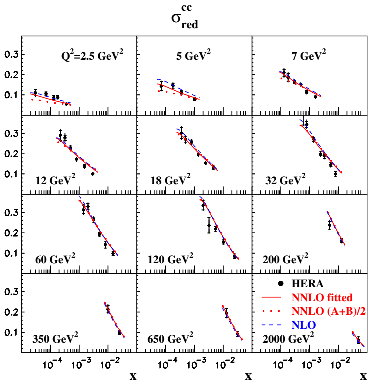
$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$$

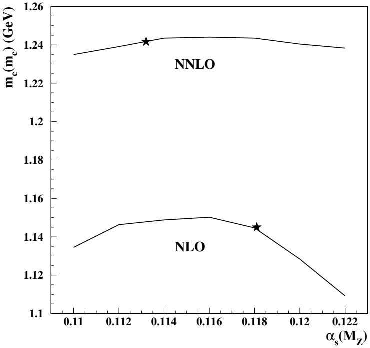
$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions.

Deep–Inelastic Scattering (DIS):





NNLO:

S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]

$$m_c(m_c) = 1.24 \pm 0.03(exp) \begin{array}{c} +0.03 \\ -0.02 \end{array} (scale) \begin{array}{c} +0.00 \\ -0.07 \end{array} (thy),$$

$$\alpha_s(M_Z^2) = 0.1132 \pm 0.011$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227].

$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses [from ABM13]

	$\alpha_s(M_Z^2)$	
BBG	$0.1134\ ^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1140 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
MSTW	$0.1155 - 0.1175$	(2013)
ABM11$_J$	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131\ ^{+\ 0.0028}_{-\ 0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141\ ^{+0.0020}_{-0.0022}$	valence analysis, N^{3}LO

$$\Delta_{\rm TH}\alpha_s = \alpha_s({\rm N^3LO}) - \alpha_s({\rm NNLO}) + \Delta_{\rm HQ} = +0.0009 \pm 0.0006_{\rm HQ}$$

NNLO accuracy is needed to analyze the world data. $\Longrightarrow$ NNLO HQ corrections needed.

Goals

- Complete the NNLO heavy flavor Wilson coefficients for twist-2 in the dynamical safe region $Q^2 > 20 GeV^2$ (no higher twist) for $F_2(x, Q^2)$
- Measure m_c and α_s as precisely as possible
- Provide precise CC heavy flavor corrections
- Consequences for LHC:
  - NNLO VFNS will be provided
  - better constraint on sea quarks and the gluon
  - precise m_c and α_s on input

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\;\delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\;\delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\;A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2 + A_{gq,Q}^{(2)}(N_F)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\Big[\;A_{qg,Q}^{(3)}(N_F+1)\;\delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+ A_{Qg}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\;, \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\;A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;\delta_2 + \;\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] + a_s^3\Big[\;A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2 \\
&+ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\;, \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\;A_{Qg}^{(1)}(N_F+1)\;\delta_2 + \;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big] + a_s^2\Big[\;A_{Qg}^{(2)}(N_F+1)\;\delta_2 \\
&+ A_{Qg}^{(1)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] + \;a_s^3\Big[\;A_{Qg}^{(3)}(N_F+1)\;\delta_2 + \;A_{Qg}^{(2)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&+ A_{gg,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \;A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\} + \;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + \;\tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
\end{aligned}
$$

J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, Nucl. Phys. B **844** (2011) 26

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2+\hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2+A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big]+a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 \\
&+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2+\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]+a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 \\
&+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2+\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
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&+A_{gg,Q}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qg}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\;, \\
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&+\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\;, \\
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&+A_{Qg}^{(1)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\;a_s^3\Big[\;A_{Qg}^{(3)}(N_F+1)\,\delta_2+\;A_{Qg}^{(2)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&+A_{gg,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\;A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\;\tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
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J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, Nucl. Phys. B **844** (2011) 26; J. Ablinger et al., 2014

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&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2+A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big]+a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 \\
&+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2+\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]+a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 \\
&+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2+\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
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$$

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Variable Flavor Number Scheme

$$
\begin{aligned}
\boxed{f_k(n_f+1,\mu^2)+f_{\overline{k}}(n_f+1,\mu^2)} &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\left[f_k(n_f,\mu^2)+f_{\overline{k}}(n_f,\mu^2)\right] \\
&\quad+\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2)+\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2)+\tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2)\;. \\
\boxed{G(n_f+1,\mu^2)} &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2)+A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2)\;. \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2)+f_{\overline{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right)+n_f\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)+\tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad\otimes\Sigma(n_f,\mu^2) \\
&\quad+\left[n_f\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)+\tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right]\otimes G(n_f,\mu^2)
\end{aligned}
$$

The choice of matching scales is not free and varies with the process in case of precision observables. Blümlein, van Neerven [hep-ph/9811351]

$\Longrightarrow$ More complicated for 2 masses J. Blümlein, Wißbrock, 2013

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009]

Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]

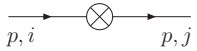
Terms $\propto n_f$ to F_2 (general N): [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011]

At 3-loop order known

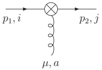
- $A_{qq,Q}^{\rm PS}$, $A_{qg,Q}$, $A_{qq,Q}^{\rm PS}$, $A_{qq,Q}^{\rm NS,\,TR}$, $A_{gq,Q}$ and $A_{Qq}^{\rm PS}$ complete
- A_{gg}: also complete (This talk)
- A_{Qg}: all terms of $O(n_f T_F^2 C_{A/F})$
- Two masses $m_1 \neq m_2$

2. Calculation of the 3-loop operator matrix elements

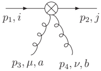
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



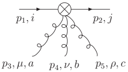
$$\delta^{ij} \Delta\!\!\!/\, \gamma_\pm (\Delta\cdot p)^{N-1}\,, \quad N\geq 1$$



$$g t^a_{ji} \Delta^\mu \Delta\!\!\!/\, \gamma_\pm \sum_{j=0}^{N-2} (\Delta\cdot p_1)^j (\Delta\cdot p_2)^{N-j-2}\,, \quad N\geq 2$$



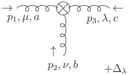
$$g^2 \Delta^\mu \Delta^\nu \Delta\!\!\!/\, \gamma_\pm \sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \Big[(t^a t^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1} + (t^b t^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\Big]\,, \quad N\geq 3$$



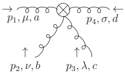
$$\begin{aligned} & g^3 \Delta_\mu\Delta_\nu\Delta_\rho \Delta\!\!\!/\, \gamma_\pm \sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2} \\ & \Big[\ (t^a t^b t^c)_{ji} (\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1} (\Delta.p_5+\Delta.p_1)^{m-l-1} \\ & +(t^a t^c t^b)_{ji} (\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1} (\Delta.p_4+\Delta.p_1)^{m-l-1} \\ & +(t^b t^a t^c)_{ji} (\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1} (\Delta.p_5+\Delta.p_1)^{m-l-1} \\ & +(t^b t^c t^a)_{ji} (\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1} (\Delta.p_3+\Delta.p_1)^{m-l-1} \\ & +(t^c t^a t^b)_{ji} (\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1} (\Delta.p_4+\Delta.p_1)^{m-l-1} \\ & +(t^c t^b t^a)_{ji} (\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1} (\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\,, \\ & N\geq 4 \end{aligned}$$

$$\gamma_+ = 1\,, \quad \gamma_- = \gamma_5\,.$$

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2} \Big[g_{\mu\nu}(\Delta\cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu)\Delta\cdot p + p^2\Delta_\mu\Delta_\nu\Big]\,, \quad N\geq 2$$

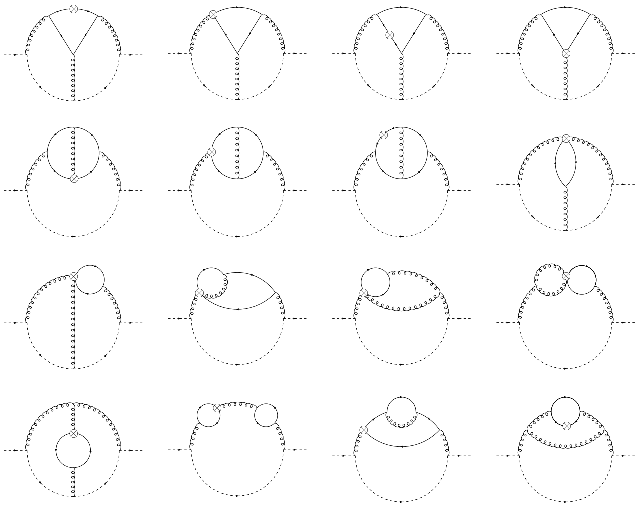


$$\begin{aligned} & -ig\frac{1+(-1)^N}{2} f^{abc}\Bigg(\\ & \Big[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu})\Delta\cdot p_1 + \Delta_\mu(p_{1,\nu}\Delta_\lambda - p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2} \\ & +\Delta_\lambda\Big[\Delta\cdot p_1 p_{2,\mu}\Delta_\nu + \Delta\cdot p_2 p_{1,\nu}\Delta_\mu - \Delta\cdot p_1\Delta\cdot p_2 g_{\mu\nu} - p_1\cdot p_2\Delta_\mu\Delta_\nu\Big] \\ & \times\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j} \\ & +\left\{ {p_1\to p_2\to p_3\to p_1 \atop \mu\to\nu\to\lambda\to\mu} \right\} + \left\{ {p_1\to p_3\to p_2\to p_1 \atop \mu\to\lambda\to\nu\to\mu} \right\}\Bigg)\,, \quad N\geq 2 \end{aligned}$$



$$\begin{aligned} & g^2\frac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4) \\ & +f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4) + f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\,, \\ & O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4) = \Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2} \\ & +[p_{4,\mu}\Delta_\sigma - \Delta\cdot p_4 g_{\mu\sigma}]\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i} \\ & -[p_{1,\sigma}\Delta_\mu - \Delta\cdot p_1 g_{\mu\sigma}]\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i} \\ & +[\Delta\cdot p_1\Delta\cdot p_4 g_{\mu\sigma} + p_1\cdot p_4\Delta_\mu\Delta_\sigma - \Delta\cdot p_4 p_{1,\sigma}\Delta_\mu - \Delta\cdot p_1 p_{4,\mu}\Delta_\sigma] \\ & \times\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\} \\ & -\left\{ {p_1\leftrightarrow p_2 \atop \mu\leftrightarrow\nu} \right\} - \left\{ {p_3\leftrightarrow p_4 \atop \lambda\leftrightarrow\sigma} \right\} + \left\{ {p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4 \atop \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma} \right\}\,, \quad N\geq 2 \end{aligned}$$

Sample of diagrams (for $A_{Qq}^{(3)\,\mathrm{PS}}$)



The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	123	642	1233

A Form program was written in order to perform the γ-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$
\begin{aligned}
A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow \text{7426 scalar integrals.} \\
A_{gq,Q}^{(3)} &\rightarrow \text{12529 scalar integrals.} \\
A_{Qq}^{(3),\mathrm{PS}} &\rightarrow \text{5470 scalar integrals.}
\end{aligned}
$$

$\Longrightarrow$ Need to use integration by parts identities.

Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta's algorithm. It is somewhat difficult to adapt this algorithm to the case where we have operator insertions, due to the dependance on the arbitrary parameter N. For this reason we apply the following trick:

$$(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

This can be then treated as an additional propagator, and Laporta's algorithm can be applied without further modification.

If we denote the master integrals by M_i, then the reduction algorithm will allow us to express any given integral I as

$$I = \sum_i c_i(x) M_i(x)$$

In fact, any given diagram D will be written this way: $D = \sum_i c_i(x) M_i(x)$

In general the coefficients $c_i(x)$ will be rational functions of x, m^2, $\Delta.p$ and the dimension D.

We want to obtain each diagram $D(N)$ as a function of N. We proceed as follows:

1. Calculate the master integrals $M_i(N)$ as functions of N.
2. Evaluate $M_i(x) = \sum_{N=0}^{\infty} x^N M_i(N)$.
3. Insert the results in $D(x) = \sum_i c_i(x) M_i(x)$.
4. Obtain $D(N)$ by extracting the Nth term in the Taylor expansion of $D(x)$.

Step 1 is done using a variety of techniques to be described shortly.

Steps 2 to 4 are done using the Mathematica packages "HarmonicSums.m", "SumProduction.m" and "EvaluateMultiSums.m" by J. Ablinger and C. Schneider.

Number of master integrals:

$$\begin{aligned}
A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow \quad 35 \ \text{ master integrals.} \\
A_{gq,Q}^{(3)} &\rightarrow \quad 41 \ \text{ master integrals.} \\
A_{Qq}^{(3),\mathrm{PS}} &\rightarrow \quad 66 \ \text{ master integrals.} \\
A_{gg,Q}^{(3)} &\rightarrow \quad 205 \ \text{ master integrals.} \\
A_{Qg}^{(3)} &\rightarrow \quad 334 \ \text{ master integrals.}
\end{aligned}$$

24 integral families are required and implemented in Reduze.

Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

- Hypergeometric functions.
- Symbolic summation algorithms based on difference fields, implemented in the `Mathematica` program `Sigma` [C. Schneider, 2005–].
- Mellin-Barnes representations.
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- Almkvist-Zeilberger algorithm.
- Differential (difference) equations.

Differential (difference) equations

In general our N-dependent Feynman master integrals will have the following structure

$$M(N) = F(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N$$

Differentiating the master integrals w.r.t. m^2 or $\Delta \cdot p$ doesn't give any new information, since we already know the functional dependence on these invariants. Differentiating w.r.t. N makes no sense.

However, in the x representation of the integrals

$$M(x) = \sum_{N=0}^{\infty} x^N F(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N \,,$$

we can differentiate w.r.t. x, since as we saw before, this turns the operator insertions into artificial propagators.

Differentiation will raise the powers of the artificial propagators and the resulting integrals can be re-expressed in terms of master integrals.

$$\frac{d}{dx} M(x) = \sum_i \frac{p_i(x)}{q_i(x)} M_i(x)$$

Integrals in a given sector will produce a system of coupled differential equations.

Let's consider the following example,

$$\begin{aligned} M_1(N) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{\sum_{j=0}^{N} (\Delta \cdot k_3)^j (\Delta \cdot k_3 - \Delta \cdot k_1)^{N-j}}{D_1 D_2 D_3 D_4 D_5}, \\ M_2(N) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{\sum_{j=0}^{N} (\Delta \cdot k_3)^j (\Delta \cdot k_3 - \Delta \cdot k_1)^{N-j}}{D_1^2 D_2 D_3 D_4 D_5}, \end{aligned}$$

where $D_1 = (k_1 - p)^2$, $D_2 = (k_2 - p)^2$, $D_3 = k_3^2 - m^2$, $D_4 = (k_3 - k_1)^2 - m^2$ and $D_5 = (k_3 - k_2)^2 - m^2$.

In the x representation, they become

$$\begin{aligned} M_1(x) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6 D_7}, \\ M_2(x) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{1}{D_1^2 D_2 D_3 D_4 D_5 D_6 D_7}, \end{aligned}$$

with $D_6 = 1 - x\Delta \cdot k_3$ and $D_7 = 1 - x(\Delta \cdot k_3 - \Delta \cdot k_1)$. So,

$$\begin{aligned} \frac{d}{dx} M_1(x) &= \frac{1}{1-x}\left(2+\epsilon-\frac{1}{x}\right) M_1(x) + \frac{2x}{1-x} M_2(x) + K_1(x) \\ \frac{d}{dx} M_2(x) &= -\frac{1}{1-x}\left(\frac{1-2\epsilon}{x} + \frac{3}{2}\epsilon - 2\right) M_2(x) \\ &\quad + \frac{\epsilon}{4}(2+3\epsilon)\frac{1}{1-x}\left(\frac{1}{x^2} - \frac{1}{x}\right) M_1(x) + K_2(x) \end{aligned}$$

where $K_1(x)$ and $K_2(x)$ are linear combinations of (already solved) subsector master integrals.

$K_1(x)$ and $K_2(x)$ can be turned into the N representation using the RISC Mathematica packages. Then using the fact that

$$
\begin{aligned}
M_1(x) &= \sum_{N=0}^{\infty} x^N F_1(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N \,, \\
M_2(x) &= \sum_{N=0}^{\infty} x^N F_2(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N \,,
\end{aligned}
$$

we get the following system of coupled difference equations:

$$
\begin{aligned}
&(N+2)F_1(N+1) - (N+2+\epsilon)F_1(N) - 2F_2(N-1) = K_1(N)\,, \\
&(N+2-2\epsilon)F_2(N+1) - \left(N+2-\frac{3}{2}\epsilon\right) F_1(N) \\
&-\frac{\epsilon}{4}(2+3\epsilon)\left(F_1(N+2) - F_1(N+1)\right) = K_2(N)\,,
\end{aligned}
$$

The system can be solved using the RISC Mathematica packages (C. Schneider's talk).

The results are given in terms of standard harmonic sums:

$$
\begin{aligned}
F_1(N) \quad = \quad & \frac{1}{\epsilon^3}\left[-\frac{8}{3}S_{-2}(N)-\frac{4}{3}S_2(N)+\frac{4N}{3(N+1)^2}+\frac{8(-1)^N(N+2)}{3(N+1)^2}\right]+\frac{1}{\epsilon^2}\left[(-1)^N\left(\frac{4(N-1)S_1(N)}{3(N+1)^2}\right.\right.\\
& \left.-\frac{4\left(5N^2+12N+9\right)}{3(N+1)^3}\right)+S_{-2}(N)\left(\frac{4(3N+1)}{3(N+1)}-\frac{8}{3}S_1(N)\right)+\frac{2(3N+1)S_2(N)}{3(N+1)}-\frac{4}{3}S_1(N)S_2(N)+\frac{2}{3}S_3(N)\\
& \left.+\frac{16}{3}S_{-2,1}(N)+\frac{4}{3}S_{2,1}(N)-\frac{2\left(5N^2+10N+4\right)}{3(N+1)^3}\right]+\frac{1}{\epsilon}\left[\frac{\left(-9N^2-10N-7\right)S_2(N)}{3(N+1)^2}\right.\\
& +(-1)^N\left(\frac{(N-1)S_1^2(N)}{3(N+1)^2}-\frac{2N(5N+9)S_1(N)}{3(N+1)^3}+\frac{(7N+5)S_2(N)}{3(N+1)^2}+\frac{2\left(19N^3+61N^2+68N+30\right)}{3(N+1)^4}\right)\\
& +\zeta_2\left(-S_{-2}(N)-\frac{1}{2}S_2(N)+\frac{N}{2(N+1)^2}+\frac{(-1)^N(N+2)}{(N+1)^2}\right)-\frac{2}{3}S_2(N)S_1^2(N)+\left(\frac{2(3N+1)S_2(N)}{3(N+1)}-\frac{10}{3}S_3(N)\right.\\
& \left.+\frac{4}{3}S_{-2,1}(N)+\frac{4}{3}S_{2,1}(N)\right)S_1(N)-\frac{1}{3}S_2^2(N)+S_{-3}(N)\left(\frac{2N}{N+1}-2S_1(N)\right)+S_{-2}(N)\left(-\frac{4}{3}S_1^2(N)\right.\\
& \left.+\frac{4(3N+1)S_1(N)}{3(N+1)}+\frac{4}{3}S_2(N)-\frac{2(9N+7)}{3(N+1)}\right)+\frac{(9N-1)S_3(N)}{3(N+1)}+\frac{2}{3}S_4(N)-2S_{-3,1}(N)-\frac{4(3N+2)S_{-2,1}(N)}{3(N+1)}\\
& \left.-\frac{4}{3}S_{-2,2}(N)-\frac{2(3N+1)S_{2,1}(N)}{3(N+1)}+\frac{10}{3}S_{3,1}(N)+\frac{4}{3}S_{-2,1,1}(N)-\frac{4}{3}S_{2,1,1}(N)+\frac{19N^3+63N^2+69N+24}{3(N+1)^4}\right]\\
& -\frac{2}{9}S_2(N)S_1^3(N)-2S_{2,2,1}(N)-\frac{16}{3}S_{3,1,1}(N)-\frac{14}{3}S_{-2,1,1,1}(N)+\frac{4}{3}S_{2,1,1,1}(N)+\frac{8}{3}S_{-2,2,1}(N)\\
& +\frac{14}{3}S_{2,1,-2}(N)+\frac{2(3N+1)S_{2,1,1}(N)}{3(N+1)}+\frac{25}{3}S_{4,1}(N)+\cdots
\end{aligned}
$$

More complicated integrals produce generalized sums, such as the cyclotomic harmonic sums:

$$S_{\{a_1,b_1,c_1\},\ldots\{a_l,b_l,c_l\}}(s_1,\ldots,s_l,N) = \sum_{k_1=1}^{N} \frac{s_1^k}{(a_1 k_1 + b_1)^{c_1}} S_{\{a_2,b_2,c_2\},\ldots\{a_l,b_l,c_l\}}(s_2,\ldots,s_l,N)$$

and sums involving inverse binomials, for example,

$$\binom{2N}{N} \sum_{k=0}^{N} \frac{k^4 2^k}{\binom{2k}{k}} S_1(k)$$

or

$$\sum_{i=1}^{N} \binom{2i}{i} (-2)^i \sum_{j=1}^{i} \frac{1}{j\binom{2j}{j}} S_{1,2}\left(\tfrac{1}{2},-1;j\right)$$

Emergence of new nested sums :

$$\sum_{i=1}^{N} \binom{2i}{i} (-2)^i \sum_{j=1}^{i} \frac{1}{j\binom{2j}{j}} S_{1,2}\left(\tfrac{1}{2}, -1; j\right)$$
$$= \int_0^1 dx \frac{x^N - 1}{x - 1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{w_{17},-1,0}(x) - 2\mathrm{H}^*_{w_{18},-1,0}(x)\right]$$
$$+ \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^N - 1}{x+1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{12}(x) - 2\mathrm{H}^*_{13}(x)\right]$$
$$+ c_3 \int_0^1 dx \frac{(-8x)^N - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1-x}} \,,$$

$$w_{12} = \frac{1}{\sqrt{x(8-x)}}, \qquad w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}},$$
$$w_{17} = \frac{1}{\sqrt{x(8+x)}}, \qquad w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}} \,.$$

$\sim$ 100 associated independent nested sums. The associated iterated integrals request root-valued alphabets with about 30 new letters. J. Ablinger, J. Bümlein, J. Raab, C. Schneider 2014.

Results

By now we have computed 6 out of 7 OMEs at $O(\alpha_s^3)$, namely,

$A_{qq}^{(3),\mathrm{PS}}$, $A_{qg}^{(3)}$ [Ablinger, Blümlein, Klein, Schneider, Wissbrock, arXiv:1008.3347]

$A_{qq}^{(3),\mathrm{NS},\,\mathrm{TR}}$ [Ablinger, Behring, Blümlein, ADF, von Manteuffel, Schneider, *et. al.*, arXiv:1406.4654]

$A_{gq}^{(3)}$ [Ablinger, Blümlein, ADF, von Manteuffel, Schneider, *et. al.*, arXiv:1402.0359]

$A_{Qq}^{(3),\mathrm{PS}}$ [Ablinger, Behring, Blümlein, ADF, von Manteuffel, Schneider, arXiv:1409.1135]

$A_{gg}^{(3)}$ to be published soon

and the corresponding Wilson coefficients:

$$L_{q,2}^{\mathrm{PS}}, \quad L_{g,2}^{\mathrm{S}}, \quad L_{q,2}^{\mathrm{NS}}, \quad \text{and} \quad H_{q,2}^{\mathrm{PS}}$$

We have also partial results for $A_{Qg}^{(3)}$ (terms $\propto N_F T_F^2$ and Ladder diagrams).

The logarithmic contributions to all OMEs and WCs were published recently [Behring, Blümlein, ADF, Bierembaum, Klein, Wißbrock, arXiv:1403.6356]

3-Loop OME: A_{Qq}^{PS}

$$
\begin{aligned}
a_{Qq}^{(3),\mathrm{PS}}(N) \quad = \quad & C_F^2 T_F \Bigg\{ \frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_{2,2}\left(2, \frac{1}{2}\right) - \frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_{3,1}\left(2, \frac{1}{2}\right) \\
& +2^N \Bigg[- \frac{32 P_3 S_{2,1}\left(1, \frac{1}{2}, N\right)}{(N-1)^2 N^3 (N+1)^2 (N+2)} - \frac{32 P_3 S_{1,1,1}\left(\frac{1}{2}, 1, 1, N\right)}{(N-1)^2 N^3 (N+1)^2 (N+2)} + \frac{32 P_4 S_{1,1}\left(1, \frac{1}{2}, N\right)}{(N-1)^3 N^4 (N+1)^2 (N+2)} + \cdots \Bigg] \\
& +2^{-N} \Bigg[- \frac{64(N^2+N+2)^2 S_{1,2,1}\left(2, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \frac{64(N^2+N+2)^2 S_{1,2,1}\left(2, 1, \frac{1}{2}, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \cdots \Bigg] + \cdots \Bigg\} \\
& + C_F T_F^2 N_F \Bigg\{ - \frac{16(N^2+N+2)^2 S_1(N)^3}{27(N-1)N^2(N+1)^2(N+2)} + \frac{16 P_9 S_1(N)^2}{27(N-1)N^3(N+1)^3(N+2)^2} \\
& + \Bigg[- \frac{208(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} S_2 - \frac{32 P_{23}}{81(N-1)N^4(N+1)^4(N+2)^3} \Bigg] S_1 \\
& + \frac{32 P_{31}}{243(N-1)N^5(N+1)^5(N+2)^4} + \frac{224(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} \zeta_3 + \cdots \Bigg\} \\
& + C_F C_A T_F \Bigg\{ \frac{2(N^2+N+2)^2 S_1(N)^4}{9(N-1)N^2(N+1)^2(N+2)} + \frac{4(N^2+N+2) P_6 S_1(N)^3}{27(N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& +2^{-N} \Bigg[\frac{16 P_2 S_3(2, N)}{(N-1)N^3(N+1)^2} - \frac{16 P_2 S_{1,2}(2,1,N)}{(N-1)N^3(N+1)^2} + \frac{16 P_2 S_{2,1}(2,1,N)}{(N-1)N^3(N+1)^2} - \frac{16 P_2 S_{1,1,1}(2,1,1,N)}{(N-1)N^3(N+1)^2} \Bigg] \\
& - \frac{32(N^2+N+2)^2 S_{1,1,2}\left(2, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \frac{32(N^2+N+2)^2 S_{1,1,2}\left(2, 1, \frac{1}{2}, N\right)}{(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32(N^2+N+2)^2 S_{1,2,1}\left(2, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 S_{1,2,1}\left(2, 1, \frac{1}{2}, N\right)}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{32(N^2+N+2)^2 S_{1,1,1,1}\left(2, \frac{1}{2}, 1, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 S_{1,1,1,1}\left(2, 1, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \cdots \Bigg\} + \cdots
\end{aligned}
$$

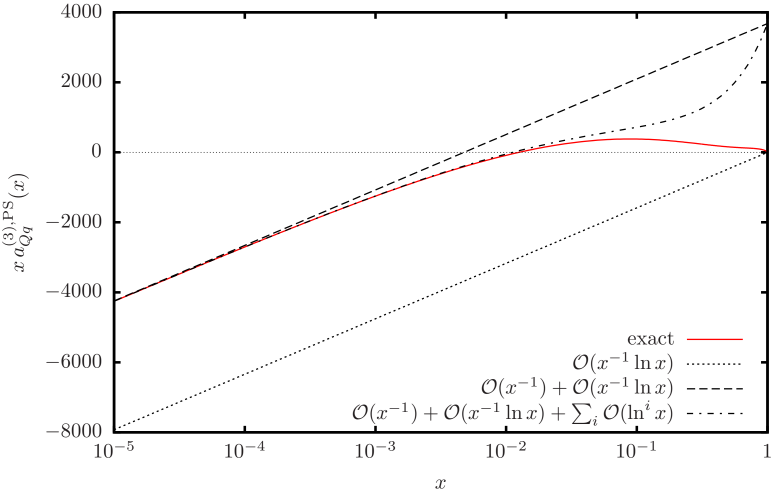


Figure: $xa_{Qq}^{(3),\mathrm{PS}}(x)$ in the low x region (solid red line) and leading terms approximating this quantity; dotted line: 'leading' small x approximation $O(\ln(x)/x)$, dashed line: adding the $O(1/x)$-term, dash-dotted line: adding all other logarithmic contributions.

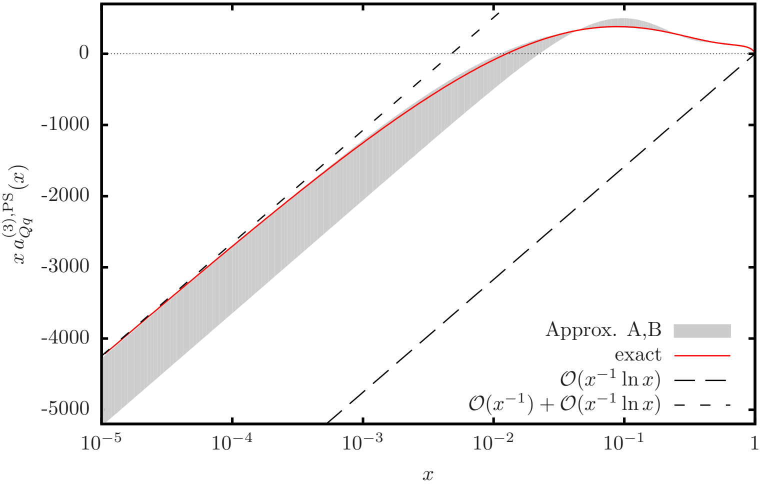


Figure: Comparison of the exact result for $xa_{Qq}^{3,\mathrm{PS}}(x)$ (solid red line) with an estimate in Kawamura et. al. (shaded area). Also shown are the first two leading small x contributions.

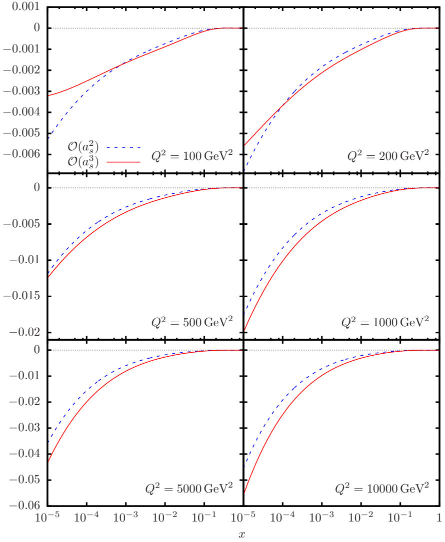


Figure: The bottom contribution by the Wilson coefficient $H_{Q,2}^{\rm PS}$ to the structure function $F_2(x, Q^2)$ as a function of x and Q^2 choosing $Q^2 = \mu^2$, $m_b = 4.78$GeV (on shell scheme).

3-Loop OME: $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = & \frac{1+(-1)^N}{2}\Bigg\{ C_F^2 T_F \Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} - \frac{4P_{69} S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& + \tilde{\gamma}_{gq}^{(0)} \left(\frac{128\left(S_{-4} - S_{-3}S_1 + S_{-3,1} + 2S_{-2,2}\right)}{3N(N+1)(N+2)} + \frac{4(5N^2+5N-22)S_1^2 S_2}{3N(N+1)(N+2)} + \dots \right) + \dots \Bigg] \\
& + C_A C_F T_F \Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{32P_2 S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{64P_{14} S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16P_{23} S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{63} S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)} + \dots \Bigg] \\
& + C_A^2 T_F \Bigg[- \frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{256P_5 S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32P_{30} S_{-2,1,1} + 16P_{35} S_{-3,1} + 16P_{44} S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52} S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36} S_2^2}{9(N-1)N^2(N+1)^2} + \dots \Bigg] \\
& + C_F T_F^2 \Bigg[- \frac{16P_{48}\binom{2N}{N} 4^{-N} \left(\sum_{i=1}^{N} \frac{4^i S_1(i-1)}{\binom{2i}{i} i^2} - 7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32P_{86} S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& + \frac{16P_{45} S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45} S_2}{9(N-1)N^3(N+1)^3(N+2)} + \dots \Bigg] + \dots \Bigg\} \qquad (1)
\end{aligned}
$$

Also, with this calculation we were able to rederive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained agreement with the literature.

Conclusions

- 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC.
  2010: Wilson Coefficients $L_q^{(3),\mathrm{PS}}(N)$, $L_g^{(3),\mathrm{S}}(N)$.
- 2013:
  - Reduze 2 has been adapted to the case where we have operator insertions, and used for reductions to master integrals.
  - Complicated master integrals have been computed using a variety of methods. In particular, the differential equations method has allowed us to tackle integrals up until now untreatable with other methods.
  - Emergence of generalized sums and new functions including a large number of root-letters in iterated integrals.
  - $A_{gq,Q}^{(3),\mathrm{S}}$, $A_{qq,Q}^{(3),\mathrm{NS,TR}}$ and $A_{Qq}^{(3),\mathrm{PS}}$ have been completed.
  - The corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio.
- 2014: The terms $\propto T_F^2$ in $A_{gg}^{(3)}$ was computed.
- 2015: Full $A_{gg}^{(3)}$ is now available.
- Different new Computer-algebra and mathematical technologies have been and continue to be developed.