Radiative corrections in bound states

RADCOR-LoopFest
Los Angeles
June 15, 2015

Andrzej Czarnecki University of Alberta

Outline

Three contexts of RadCors in bound states:

Spectrum: Lamb shift, Rydberg, proton radius

Interaction with external fields:
g-2 of a bound electron

Decay of a bound particle:
muon decay in orbit and
the muon --> electron conversion

In the beginning there was a bound state

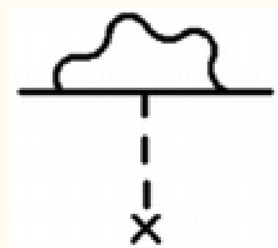
At least in the beginning of radiative corrections:

PHYSICAL REVIEW VOLUME 72, NUMBER 3 AUGUST 1, 1947

Fine Structure of the Hydrogen Atom by a Microwave Method* **

WILLIS E. LAMB, JR. AND ROBERT C. RETHERFORD
Columbia Radiation Laboratory, Department of Physics, Columbia University, New York, New York

(Received June 18, 1947)



$$W_{ns}' = 136 \ln[K/(E_n - E_m)]$$
$$= 1040 \text{ megacycles}$$

The Mother of all Radcor-Loopfests

PHYSICAL REVIEW VOLUME 72, NUMBER 4 AUGUST 15, 1947

The Electromagnetic Shift of Energy Levels

H. A. BETHE
Cornell University, Ithaca, New York

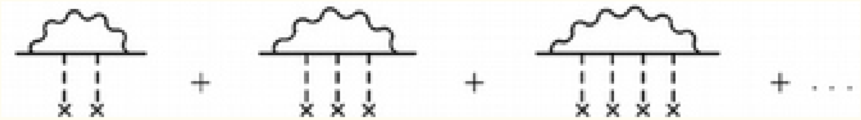
(Received June 27, 1947)

Energy of an electron in the hydrogen ground state

$$E \simeq m - \frac{m\,(Z\alpha)^2}{2}$$

Dirac eq →

$$m\sqrt{1-(Z\alpha)^2}$$



$$+\frac{\alpha}{\pi}\left[(Z\alpha)^4\,(\ln Z\alpha + c) + \ldots\right]$$

Bethe 1947
Feynman
Schwinger...

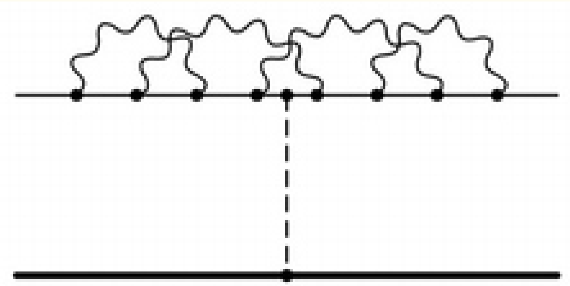
$$+\left(\frac{\alpha}{\pi}\right)^2 \ldots$$

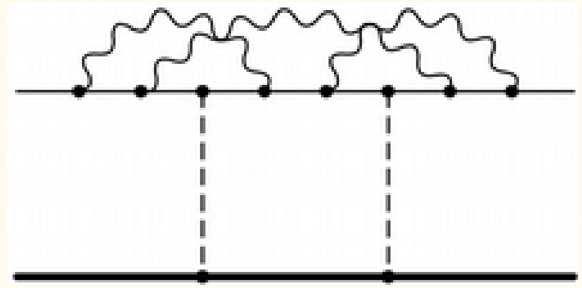
Appelquist+Brodsky
Barbieri, Magnaco, Remiddi 1970

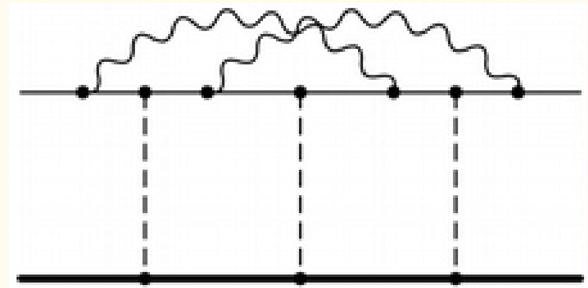
$$+\left(\frac{\alpha}{\pi}\right)^3 \ldots$$

Melnikov + van Ritbergen 1999

Now the frontier is at four loops







the same technology
as for the 4-loop g-2

enhanced

enhanced^2

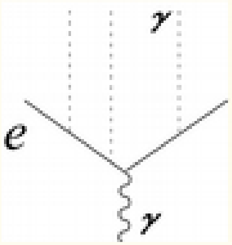
Benefits of this effort:

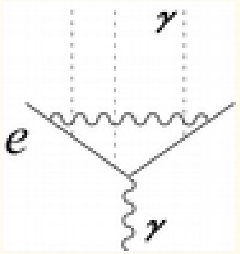
Rydberg constant: $$R_\infty = \frac{m_e c \alpha^2}{2h}$$ known to 6 ppt
second best source of alpha

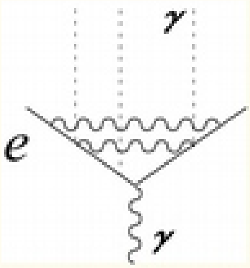
and the proton radius.

Anomalous magnetic dipole moments

Bound-electron g-2







$$g = 2 - \frac{2\,(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{6} + \ldots$$

$$+\frac{\alpha}{\pi}\left[1 + \frac{(Z\alpha)^2}{6} + (Z\alpha)^4\,(a_{41}\ln Z\alpha + a_{40}) + \ldots\right]$$

$$+\left(\frac{\alpha}{\pi}\right)^2\left[-0.65..\left(1 + \frac{(Z\alpha)^2}{6}\right) + (Z\alpha)^4\,(b_{41}\ln Z\alpha + b_{40}) + \ldots\right]$$

two-loop corrections

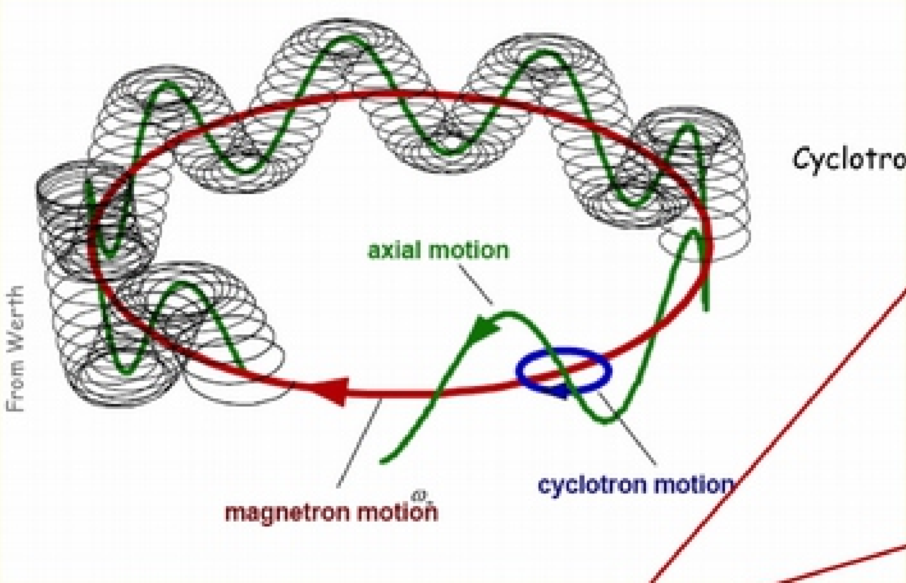
$$b_{41} = \frac{28}{9}$$

$$b_{40} = -16.4$$

Pachucki, AC Jentschura, Yerokhin

Bound g factor and the electron mass determination

Motion in a Penning trap



Spin precession (Larmor) frequency

$$\hbar\omega_L = g\frac{e}{2m}\vec{s}\cdot\vec{B}$$

Cyclotron frequency

$$\hbar\omega_c = \frac{q}{2M}\vec{L}\cdot\vec{B}$$

$$\frac{m}{M} = \frac{g\omega_c e}{2\omega_L q}$$

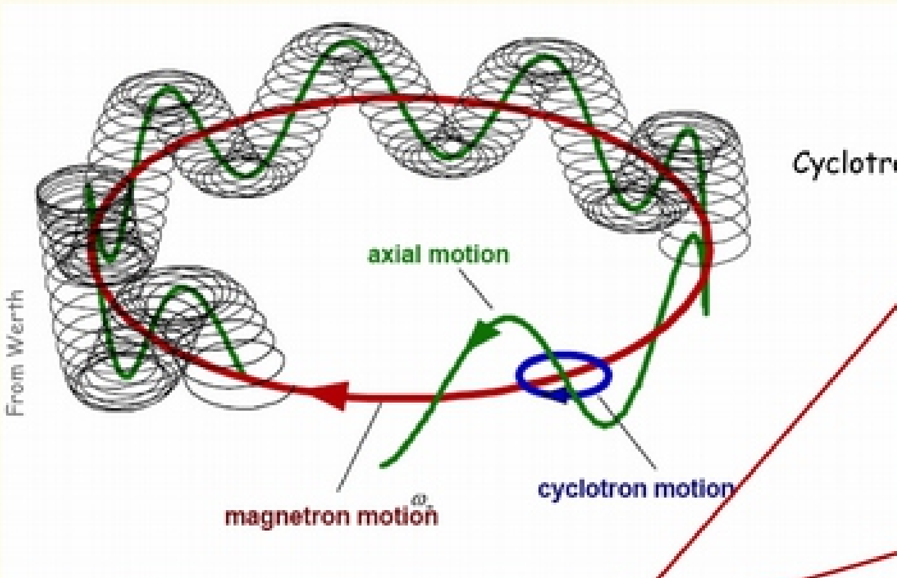
This g factor is modified
by the electron binding to the nucleus

$$m_e\left({}^{12}C^{5+}\right) = 0.000\,548\,579\,909\,31(29)_{\text{exp}}(1)_{\text{th}}\ u$$

Theoretical error: negligible

Bound g factor and the electron mass determination

Motion in a Penning trap



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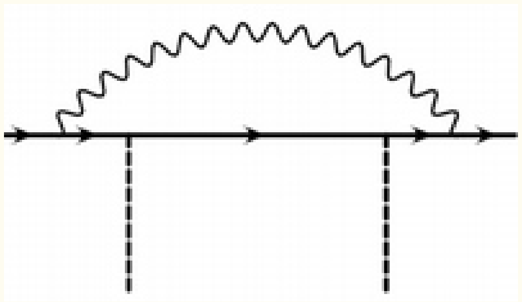
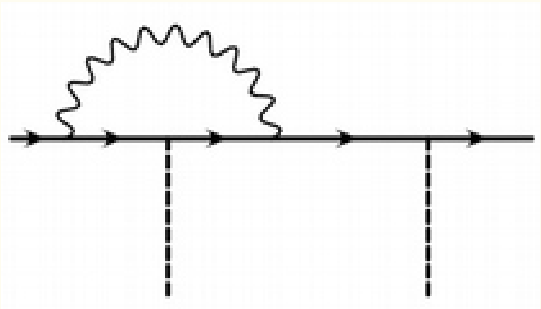
$$0.000\ 548\ 579\ 909\ 067\,(17)_{\text{exp}}$$

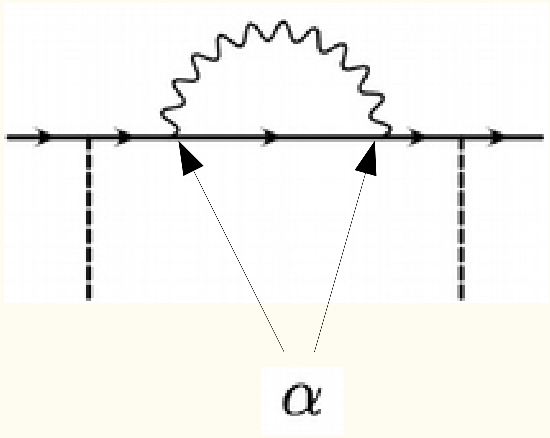
NEW Nature 2014
Sturm et al

How are the higher-order corrections to g computed?

Warm-up: Lamb shift (no external magnetic field)







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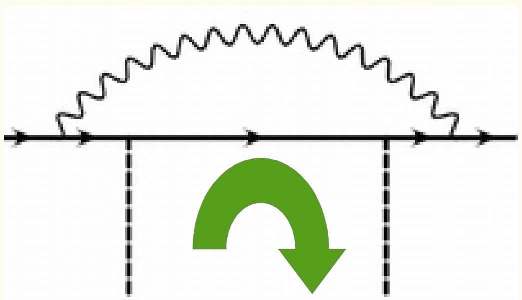
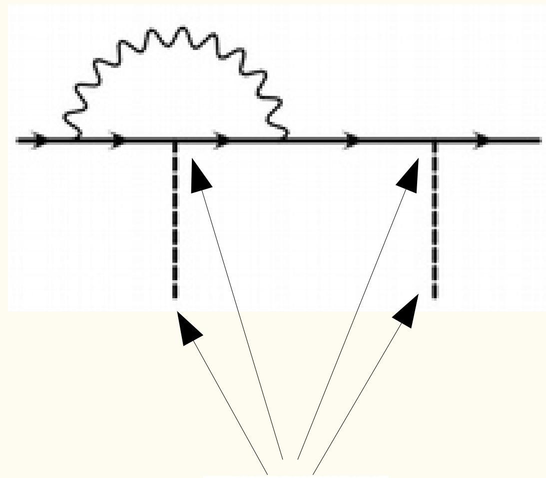
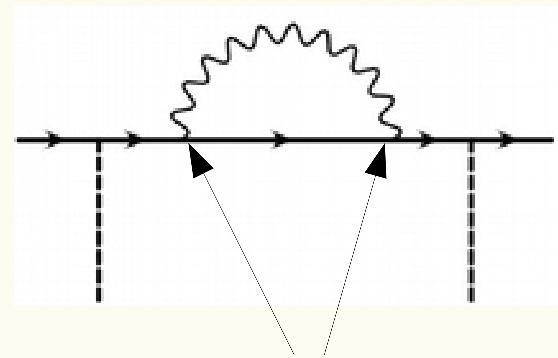


$$|\psi(r=0)|^2 \sim (Z\alpha)^3$$



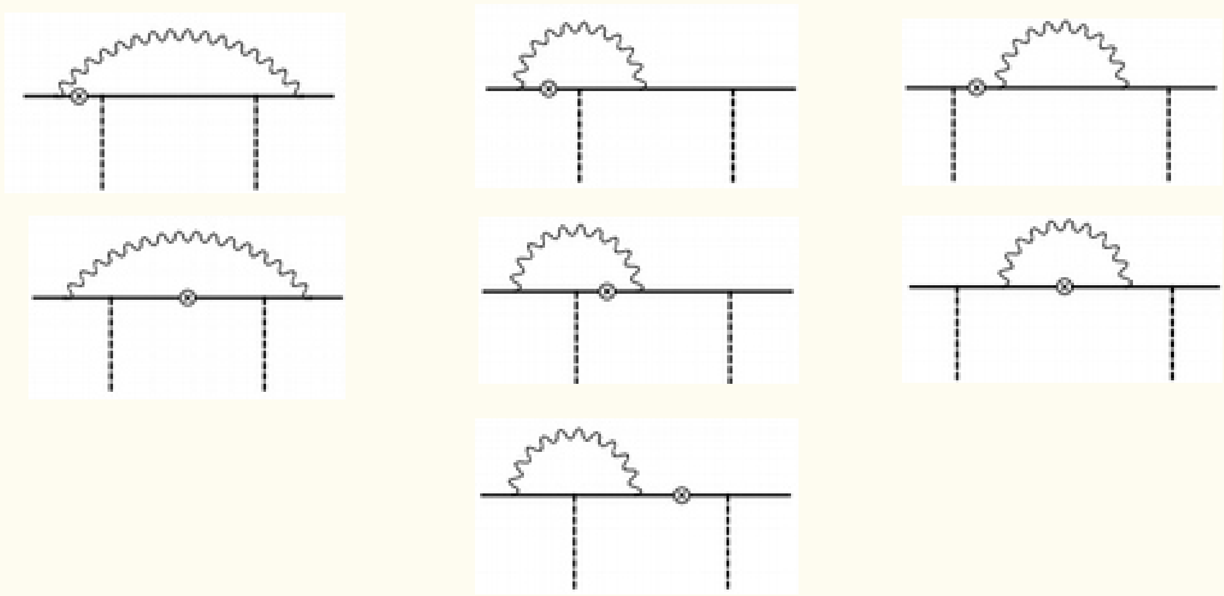
$$(Z\alpha)^2$$



$$\alpha$$

$$\Delta E \sim \alpha (Z\alpha)^5$$

To find Δg, consider the energy in a magnetic field



The result is gauge-invariant; but not yet complete.

What if the magnetic field couples to an external line?

Contribution of a short-distance perturbation to g

$$\Delta g = \frac{2\,|e|}{\mu_0 m B}\sum_{n}^{n\neq a}\frac{\langle a\,|\delta U|\,n\rangle\left\langle n\left|\vec{\alpha}\cdot\vec{A}\right|a\right\rangle}{E_a - E_n}$$

This sum can be done exactly with help of virial identities

S. Karshenboim, V. Ivanov, and V. Shabaev

$$\Delta g = \frac{2\kappa m}{j\,(j+1)}\langle n\kappa|\,\delta U\,\frac{\kappa}{m^2}\,(I - |n\kappa\rangle\langle n\kappa|)\left[\left(E_{n\kappa} - \frac{m}{2\kappa}\right) r i\sigma_y + m r\sigma_x + \alpha Z i\sigma_y - \kappa\sigma_z\right]|n\kappa\rangle$$

The result is especially simple if the short-distance perturbation is energy-independent (like the Uehling potential)

$$\Delta g = \frac{16}{3}\left(1 - \frac{1-\sqrt{1-(Z\alpha)^2}}{2(Z\alpha)^2}\right)\left\langle\delta^3(r)\right\rangle_\psi \simeq 4\left\langle\delta^3(r)\right\rangle_\psi$$

This is just Lamb

Contribution of a short-distance perturbation to g

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Slightly more work with self-energies; but we now have completed $\alpha\,(Z\alpha)^5$

Muonic atoms

200 years ago

Mass composition of air

• Let x = sp. gr. of oxygen. 22·22 = a
y = sp. gr. of azote. 77·77 = b

Then $\frac{x + 4y}{5} = 1.$

And $x : 4y :: a : b.$

Hence $5 - 4y = \frac{4ay}{b}$

And $y = \frac{5b}{4a + 4b} = \cdot 9722.$ And $x = 5 - 4y = 1\cdot 11111.$

† Let x = sp. gr. of hydrogen.

Then $\frac{3x + \cdot 9722}{2} = \cdot 5902.$

Hence $x = \frac{1\cdot 1804 - \cdot 9722}{3} = \cdot 0694.$

‡ 1·11111 ÷ ·0694 = 16. And ·9722 ÷ ·0694 = 14.

Densities of oxygen and nitrogen, relative to air

Ammonia gives hydrogen density

Integer ratios of densities O_2/H_2 and N_2/H_2

Hypothesis: elements made of hydrogen

1815: Annals of Philosophy (anonymous!)

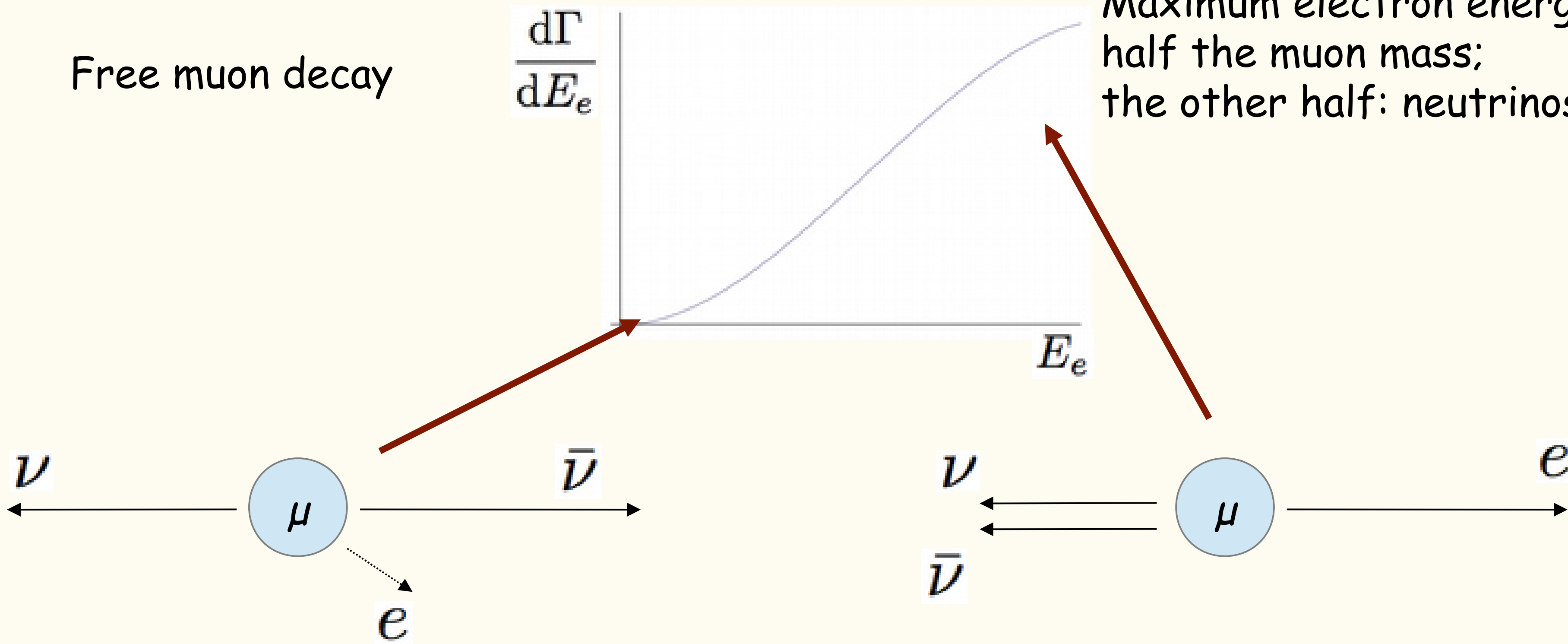
William Prout (1785–1850)

Hydrogen ..	1
Carbon	6
Azote	14
Phosphorus ..	14
Oxygen......	16
Sulphur......	16
Calcium	20
Sodium..	24
Iron	28
Zinc	32
Chlorine	36
Potassium ...	40
Barytium ...	70
Iodine	124

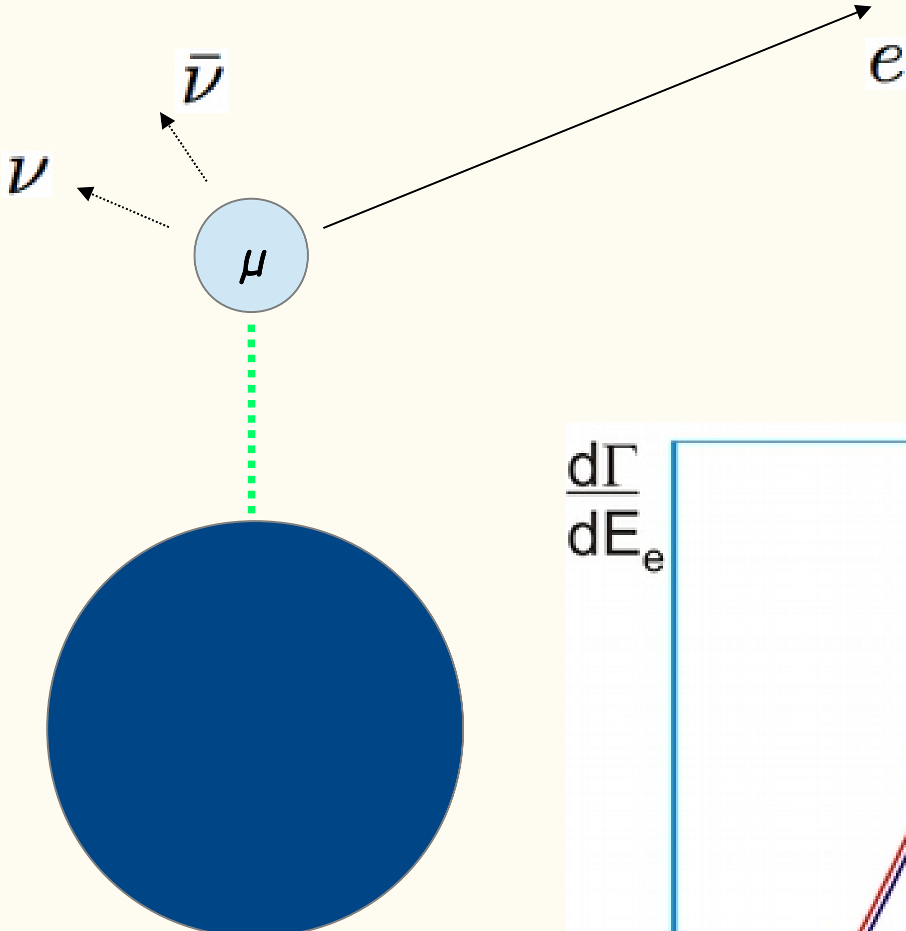
Muon decay: electron energy spectrum

Free muon decay

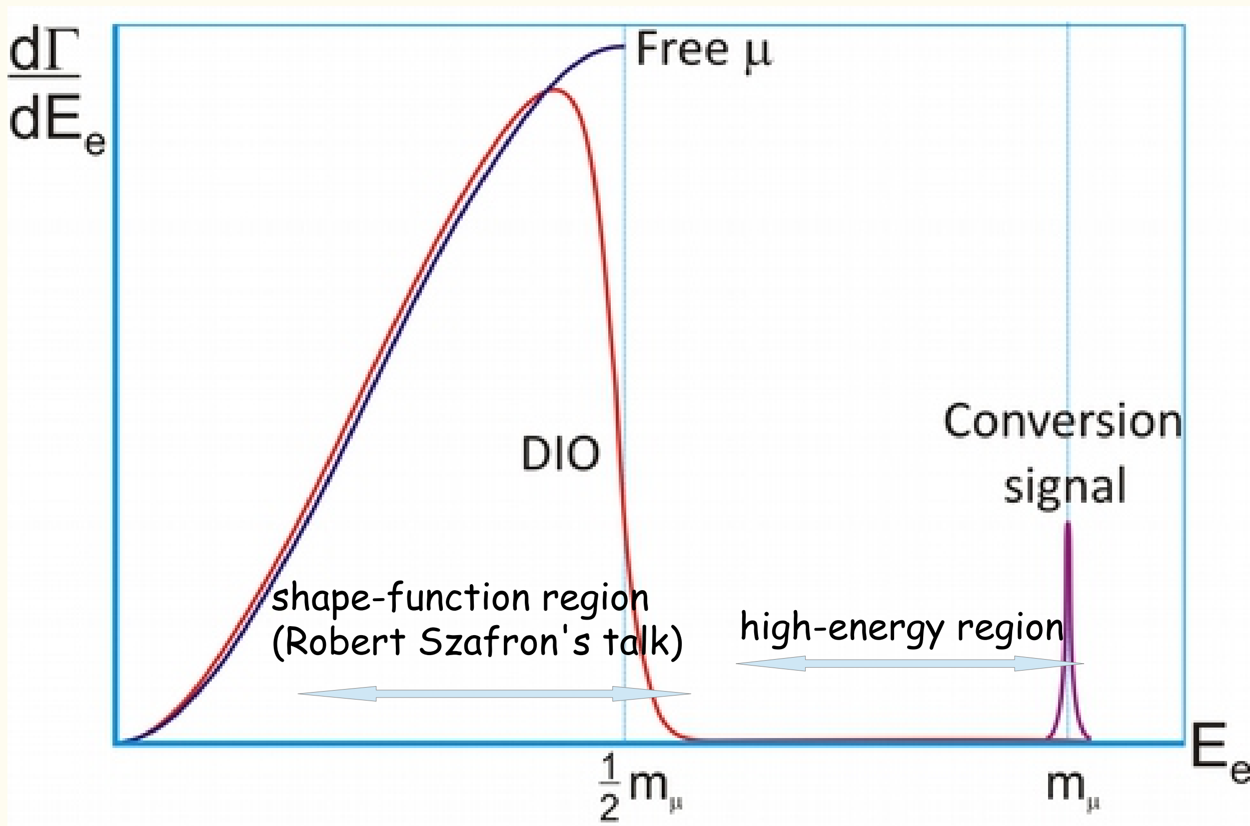
Maximum electron energy:
half the muon mass;
the other half: neutrinos.



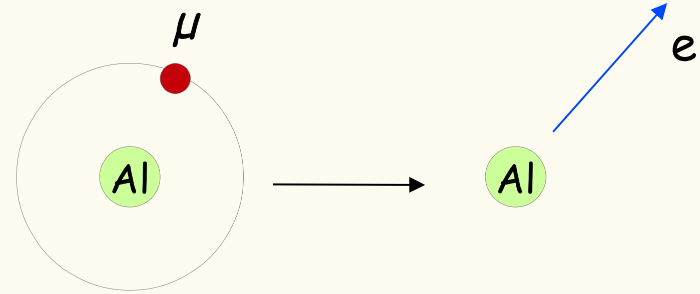
Electron spectrum in a mu-decay near nucleus



Electron energy can be as large as the whole muon mass

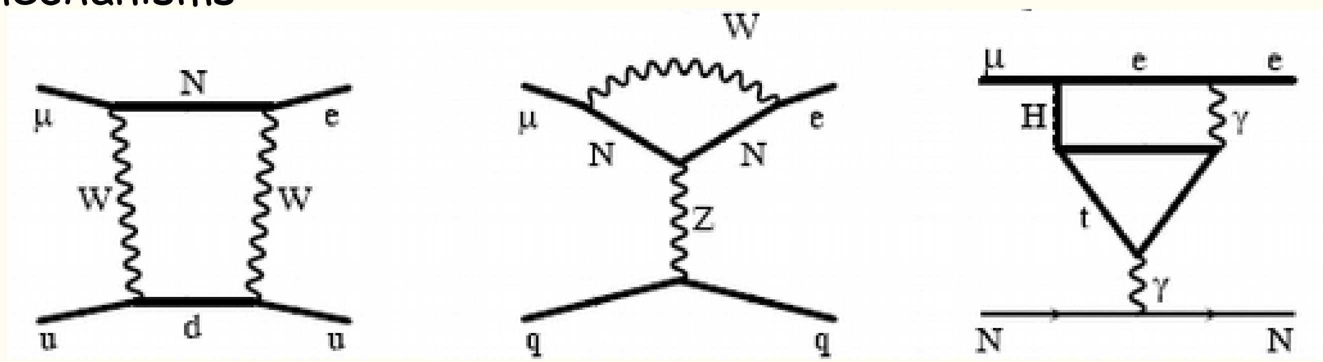


Muon-electron conversion

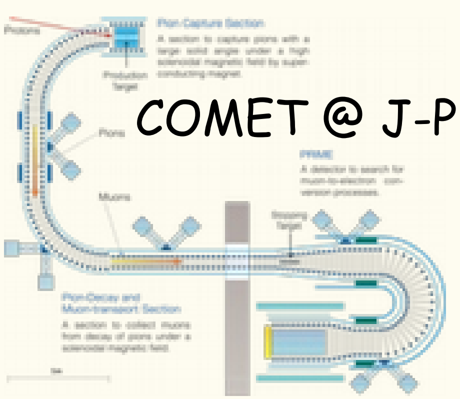


The Next Big Thing

Variety of mechanisms

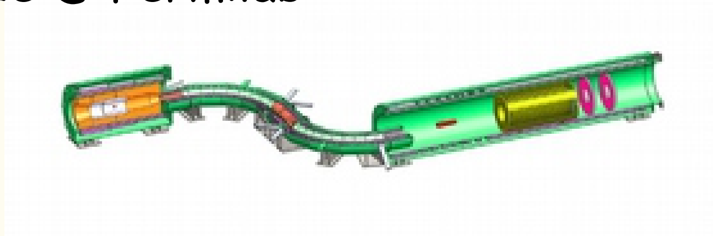


Two major new experiments

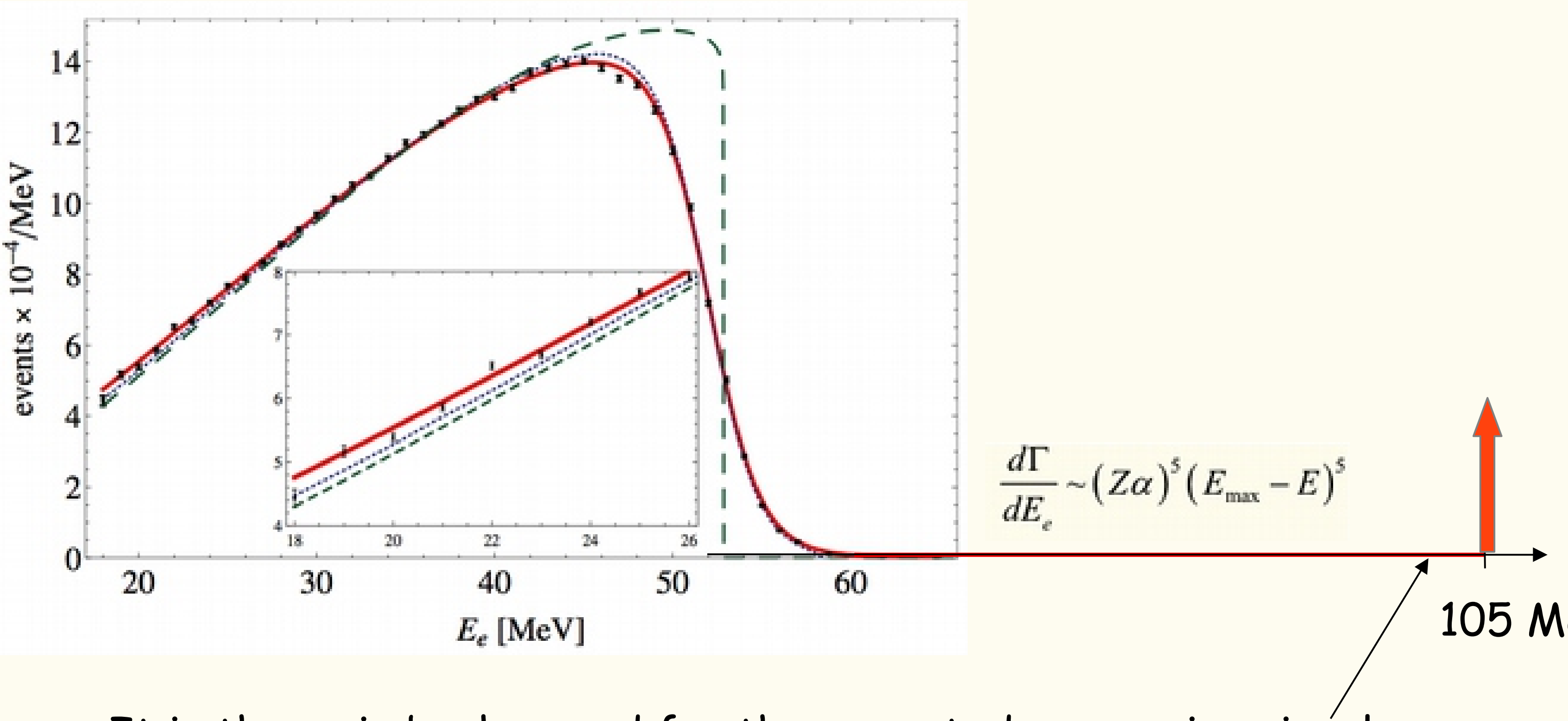


COMET @ J-PARC

Mu2e @ Fermilab



Spectrum of the bound muon decay

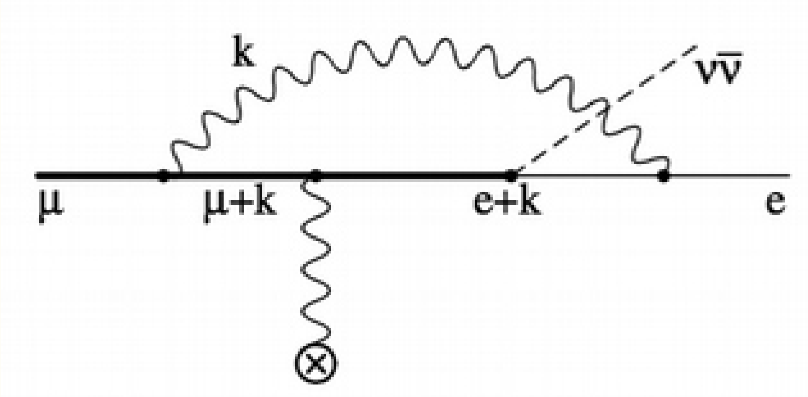


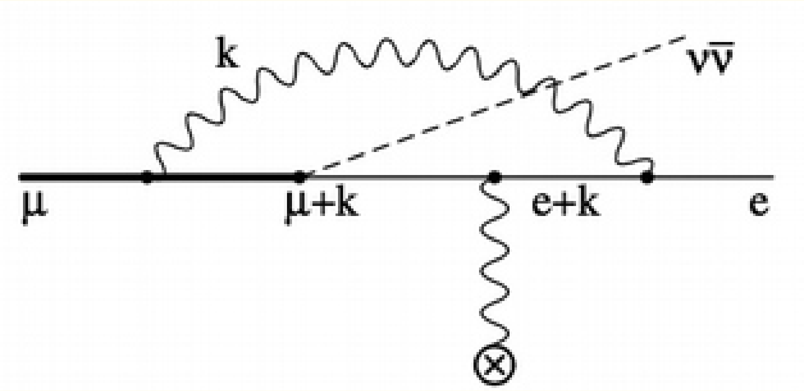
$$\frac{d\Gamma}{dE_e} \sim (Z\alpha)^5 (E_{max} - E)^5$$

It is the main background for the expected conversion signal

AC, M. Dowling, X. Garcia i Tormo, W. Marciano, R. Szafron
R. Szafron, AC

Radiative corrections to the electron spectrum





Competing effects:
- vacuum polarization in the hard photon; and
- self-energy and real radiation

$$\frac{1}{\Gamma_0}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_e}\bigg|_{E_e\to\tilde{m}} = B\Delta^5 + \mathcal{O}\left(\Delta^6\right)$$

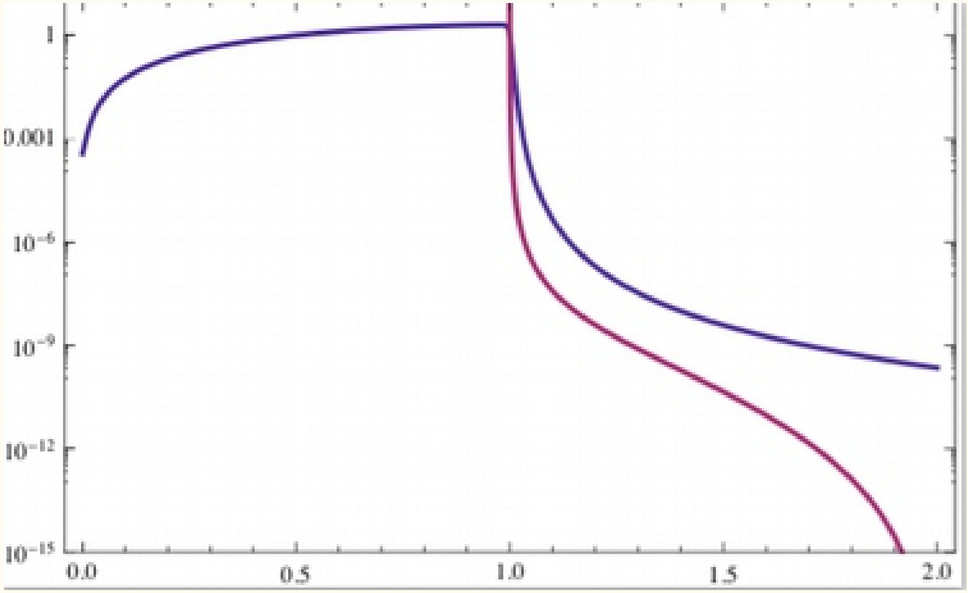
$$B|_{(Z\alpha)^5} = \frac{1024}{5\pi m_\mu^6}(Z\alpha)^5\left(\frac{\Delta}{m_\mu}\right)^{\frac{\alpha}{\pi}\delta_S}(1+\delta_{VP}+\delta_F)$$

$$\delta_S = 10.1$$

Szafron, AC arXiv:1505.05237

Ultimate goal: smooth matching of all energy regions

Status for today…



Conclusions

Many opportunities for improving bound-state theory

- hydrogen Lamb shift for a better proton radius

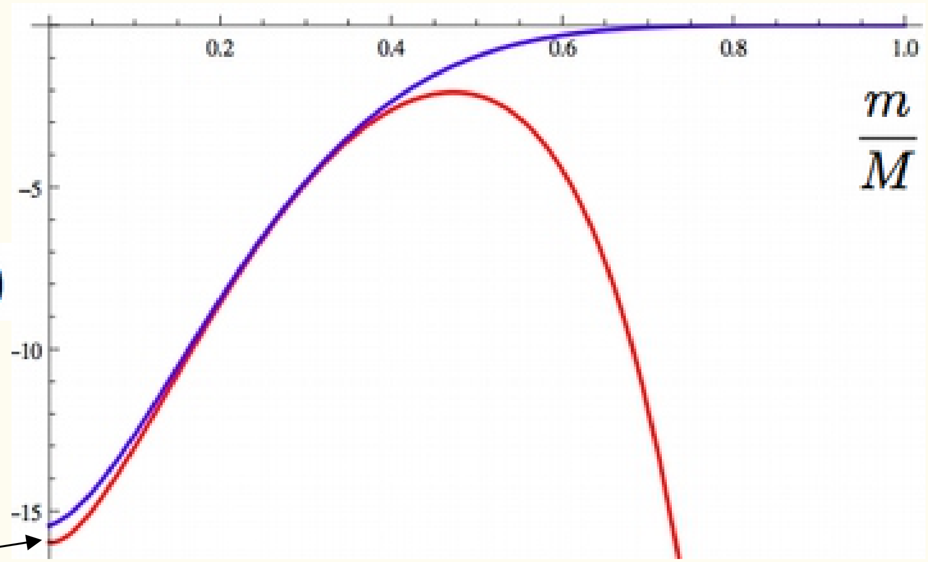
- bound electron *g*-factor: for m_e; complements a vigorous experimental program in Mainz, Heidelberg and GSI Darmstadt

- muon decay: background for μ-e conversion searches; theoretically interesting

Xtra slides

Free muon lifetime

α^2 correction to $\Gamma\left(\mu(M) \to e\,(m)\,\nu\bar{\nu}\right)$



exact

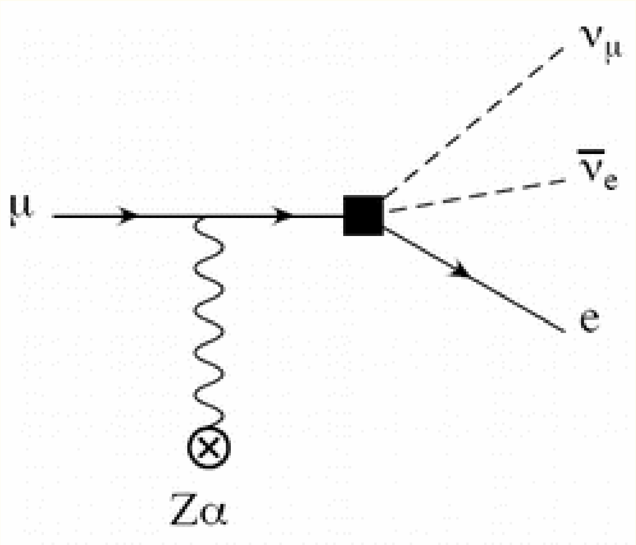
Note: the blue curve is designed for m ~ M,
but is good even for m << M.

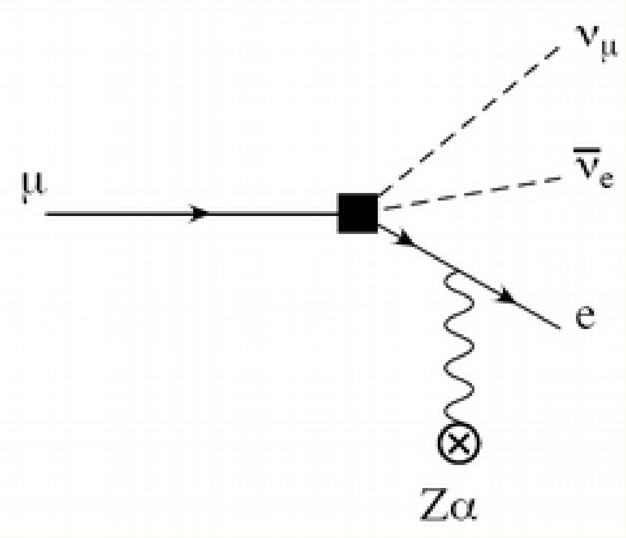
So we want to exploit the expansion around m = M to get α^3

M. Dowling, with J. Piclum, AC, M. Czakon

Origin of the $(E_{max}-E)^5$ suppression (high E)

Two diagrams with high-virtuality photons, carrying no energy:





Neutrinos get no energy;
The nucleus balances electron's momentum, takes no energy.
Near the end point:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_e} \sim |\psi(0)|^2 (Z\alpha)^2 \frac{\mathrm{d}^3\nu_e}{\nu_e}\frac{\mathrm{d}^3\nu_\mu}{\nu_\mu}\delta(E_{\max}-E_e-\nu_e-\nu_\mu)\,\mathrm{Tr}\ldots\not{\nu}_e\ldots\not{\nu}_\mu$$
$$\sim (Z\alpha)^5 (E_{\max}-E_e)^5$$