Radiative corrections in bound states

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Outline

Three contexts of RadCors in bound states:

**Spectrum**: Lamb shift, Rydberg, proton radius

**Interaction** with external fields:
  *g*-2 of a bound electron

**Decay** of a bound particle:
  muon decay in orbit and
  the muon --> electron conversion
In the beginning there was a bound state

At least in the beginning of radiative corrections:

The Mother of all Radcor-Loopfests
Energy of an electron in the hydrogen ground state

\[ E \approx m - \frac{m(Z\alpha)^2}{2} \]

\[ m\sqrt{1 - (Z\alpha)^2} \]

\[ + \frac{\alpha}{\pi} \left[ (Z\alpha)^4 (\ln Z\alpha + c) + \ldots \right] \]

\[ + \left( \frac{\alpha}{\pi} \right)^2 \ldots \]

\[ + \left( \frac{\alpha}{\pi} \right)^3 \ldots \]

Bethe 1947
Feynman
Schwinger...

Appelquist+Brodsky
Barbieri, Magnaco, Remiddi 1970

Melnikov + van Ritbergen 1999
Now the frontier is at four loops

the same technology as for the 4-loop g-2

Benefits of this effort:

Rydberg constant: $R_\infty = \frac{m_e c \alpha^2}{2\hbar}$

known to 6 ppt
second best source of alpha

and the proton radius.
Anomalous magnetic dipole moments
Bound-electron $g-2$

\[
g = 2 - \frac{2(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{6} + \ldots
\]

\[
+ \frac{\alpha}{\pi} \left[ 1 + \frac{(Z\alpha)^2}{6} + (Z\alpha)^4 (a_{41} \ln Z\alpha + a_{40}) + \ldots \right]
\]

\[
+ \left( \frac{\alpha}{\pi} \right)^2 \left[ -0.65 \left( 1 + \frac{(Z\alpha)^2}{6} \right) + (Z\alpha)^4 (b_{41} \ln Z\alpha + b_{40}) + \ldots \right]
\]

two-loop corrections

\[
b_{41} = \frac{28}{9}
\]

\[
b_{40} = -16.4
\]

Pachucki, AC
Jentschura, Yerokhin
Bound \( g \) factor and the electron mass determination

Motion in a Penning trap

\[
\hbar \omega_L = g \frac{e}{2m} \bar{s} \cdot \bar{B}
\]

\[
\hbar \omega_c = \frac{q}{2M} \bar{L} \cdot \bar{B}
\]

\[
\frac{m}{M} = \frac{g \omega_c e}{2 \omega_L q}
\]

This \( g \) factor is modified by the electron binding to the nucleus

\[
m_e(^{12}\text{C}^{5+}) = 0.000\,548\,579\,909\,31(29)_{\text{exp}}\,(1)_{\text{th}}\,u
\]

Theoretical error: negligible
Bound $g$ factor and the electron mass determination

Motion in a Penning trap

Spin precession (Larmor) frequency

\[ \hbar \omega_L = g \frac{e}{2m} \vec{s} \cdot \vec{B} \]

Cyclotron frequency

\[ \hbar \omega_c = \frac{q}{2M} \vec{L} \cdot \vec{B} \]

This $g$ factor is modified by the electron binding to the nucleus

\[ m_e\left({}^{12}\text{C}^{5+}\right) = 0.000\,548\,579\,909\,31(29)_{\text{exp}} \left(1\right)_{\text{th}} u \]

Theoretical error: negligible

0.000 548 579 909 067 (17)_{\text{exp}}

NEW Nature 2014
Sturm et al
How are the higher-order corrections to $g$ computed?

Warm-up: Lamb shift (no external magnetic field)
How are the higher-order corrections to $g$ computed?

Warm-up: Lamb shift (no external magnetic field)

\[
|\psi (r = 0)|^2 \sim (Z\alpha)^3
\]

\[
(Z\alpha)^2
\]

\[
\Delta E \sim \alpha (Z\alpha)^5
\]
To find $\Delta g$, consider the energy in a magnetic field.

The result is gauge-invariant; but not yet complete.

What if the magnetic field couples to an external line?
Contribution of a short-distance perturbation to $g$

\[
\Delta g = \frac{2|e|}{\mu_0 m B} \sum_{n \neq a} \frac{\langle n | \delta U | n \rangle \langle n | \vec{\alpha} \cdot \vec{A} | a \rangle}{E_a - E_n}
\]

This sum can be done exactly with help of virial identities.

S. Karshenboim, V. Ivanov, and V. Shabaev

The result is especially simple if the short-distance perturbation is energy-independent (like the Uehling potential):

\[
\Delta g = \frac{2km}{j(j+1)} \left( n \kappa | \delta U \frac{\kappa}{m^2} \right) \left( I - |n\kappa\rangle \langle n\kappa| \right) \left[ \left( E_{n\kappa} - \frac{m}{2\kappa} \right) r i \sigma_y + m r \sigma_x + \alpha Z i \sigma_y - \kappa \sigma_z \right] |n\kappa\rangle
\]

This is just Lamb
Contribution of a short-distance perturbation to $g$

\[
\Delta g = \frac{2 |e|}{\mu_0 m B} \sum_n \frac{n \neq a}{E_a - E_n} \langle a | \delta U | n \rangle \langle n | \vec{\alpha} \cdot \vec{A} | a \rangle
\]

This sum can be done exactly with help of virial identities

S. Karshenboim, V. Ivanov, and V. Shabaev

\[
\Delta g = \frac{2 \kappa m}{j(j+1)} \langle n\kappa | \delta U \frac{\kappa}{m^2} (I - |n\kappa\rangle \langle n\kappa|) \left[ \left( E_{n\kappa} - \frac{m}{2\kappa} \right) ri\sigma_y + m\tau\sigma_x + \alpha Zi\sigma_y - \kappa\sigma_z \right] |n\kappa\rangle
\]

The result is especially simple if the short-distance perturbation is energy-independent (like the Uehling potential)

\[
\Delta g = \frac{16}{3} \left( 1 - \frac{1 - (Z\alpha)^2}{2(Z\alpha)^2} \right) \langle \delta^3 (r) \rangle \approx 4 \langle \delta^3 (r) \rangle
\]

Slightly more work with self-energies; but we now have completed $\alpha (Z\alpha)^5$
Muonic atoms
200 years ago

Mass composition of air

Densities of oxygen and nitrogen, relative to air

Ammonia gives hydrogen density

Integer ratios of densities $\text{O}_2/\text{H}_2$ and $\text{N}_2/\text{H}_2$

Hypothesis: elements made of hydrogen

1815: Annals of Philosophy (anonymous!)
Muon decay: electron energy spectrum

Free muon decay

Maximum electron energy: half the muon mass; the other half: neutrinos.
Electron spectrum in a mu-decay near nucleus

Electron energy can be as large as the whole muon mass

shape-function region (Robert Szafron's talk)

high-energy region
Muon-electron conversion

The Next Big Thing

Variety of mechanisms

Two major new experiments

**COMET @ J-PARC**

**Mu2e @ Fermilab**
Spectrum of the bound muon decay

It is the main background for the expected conversion signal

\[ \frac{d\Gamma}{dE_e} \sim (Z\alpha)^5 (E_{\text{max}} - E)^5 \]

105 MeV
Radiative corrections to the electron spectrum

Competing effects:
- vacuum polarization in the hard photon; and
- self-energy and real radiation

\[
\frac{1}{\Gamma_0} \frac{d\Gamma}{dE_e} \bigg|_{E_e \to \tilde{m}} = B \Delta^5 + \mathcal{O}(\Delta^6)
\]

\[
B|_{(Z\alpha)^5} = \frac{1024}{5\pi m_{\mu}^6} (Z\alpha)^5 \left( \frac{\Delta}{m_{\mu}} \right)^{\alpha_\pi} \delta_S 
(1 + \delta_{VP} + \delta_F)
\]

\[
\delta_S = 10.1
\]
Ultimate goal: smooth matching of all energy regions

Status for today...
Conclusions

Many opportunities for improving bound-state theory

- hydrogen Lamb shift for a better proton radius

- bound electron g-factor: for $m_e$; complements a vigorous experimental program in Mainz, Heidelberg and GSI Darmstadt

- muon decay: background for $\mu$-e conversion searches; theoretically interesting
Xtra slides
Free muon lifetime

\[ \alpha^2 \text{ correction to } \Gamma (\mu(M) \rightarrow e(m) \nu \bar{\nu}) \]

Note: the blue curve is designed for \( m \sim M \), but is good even for \( m \ll M \).

So we want to exploit the expansion around \( m = M \) to get \( \alpha^3 \).
Origin of the \((E_{\text{max}} - E)^5\) suppression (high \(E\))

Two diagrams with high-virtuality photons, carrying no energy:

Neutrinos get no energy;
The nucleus balances electron’s momentum, takes no energy.

Near the end point:

\[
\frac{d\Gamma}{dE_e} \sim |\psi(0)|^2 (Z\alpha)^2 \frac{d^3\nu_e}{\nu_e} \frac{d^3\nu_\mu}{\nu_\mu} \delta (E_{\text{max}} - E_e - \nu_e - \nu_\mu) \text{Tr} \ldots \psi_e \ldots \psi_\mu
\]

\[
\sim \ (Z\alpha)^5 \ (E_{\text{max}} - E_e)^5
\]