Towards a four loop form factor

in progress with
Bernd Kniehl and Gang Yang

Rutger Boels
University of Hamburg

What this talk is

What this talk is

- a status report / teaser
- a cry for help
- about the wrong approach to compute an interesting quantity

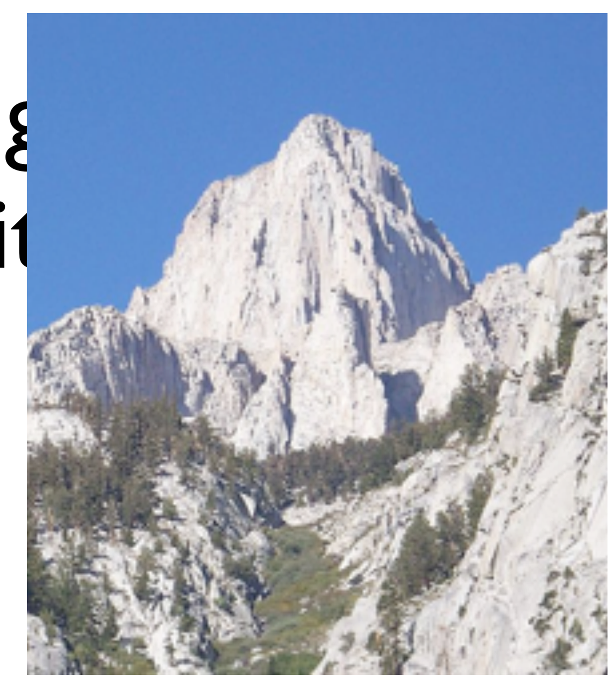
What this talk is

N=4 Yang-Mills

lightlike cusp anomalous dimension

unique maximal supersymmetric gauge theory in D=4

- about the wrong
compute an
interesting quantit



universal function in IR divergences

$$\mathcal{L} = \mathcal{L}(A_\mu, \psi^I, \phi^{[IJ]})$$

$$\gamma_{\rm cusp}(g_{\rm ym}, N_c)$$

Longer term goals

- computational overhead quickly disastrously large, both in QCD as in N=4 → techniques to combat both

three loops: [Gehrmann et.al, 06] basis of masters
[Baikov et.al, 09] first integration
[Gehrmann et.al, 10] cross-check

- N=4 ⇔ max transcendental part QCD?
- in N=4, planar limit known as solution to [Beisert-Eden-Staudacher, 04] equation (AdS/CFT, integrability)
- first non-planar correction at four loops

$$\gamma_{\rm cusp} = \sum_l g^{2l} \gamma^{(l)}_{\rm cusp} = a_1 g^2 C_A + a_2 g^4 C_A^2 + a_3 g^6 C_A^3 + g^8 \left(a_4^P C_A^4 + a_4^{NP} d_{44} \right) + \mathcal{O}(g^9)\,,$$

$$C_A = N_c \qquad d_{44} = N_c^4 + 36 N_c^2$$

- ideally, eventually, determine BES for non-planar

Wise words

"small problems at high loop orders are not small problems"

(≤ [Bern])

- color factors involve 8 structure constants (only adjoint)

$$\gamma_{\rm cusp} = \sum_l g^{2l} \gamma^{(l)}_{\rm cusp} = d_1 g^2 C_A + d_2 g^4 C_A^2 + d_3 g^6 C_A^3 + g^8 (d_4^P C_A^4 + d_4^{NP} d_{44}) + \mathcal{O}(g^9)\,,$$

- DiaGen to generate graphs,
- COLOR to compute color factors

$$C_A = N_c \qquad d_{44} = N_c^4 + 36 N_c^2$$

(works to 8 loops)

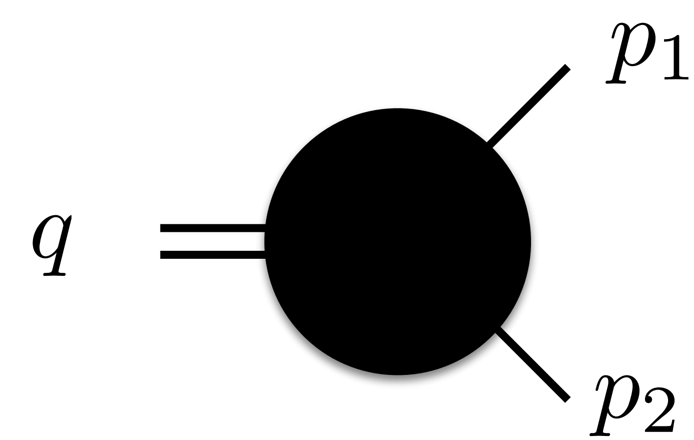
Cusp from Sudakov form factor

- cusp is universal → can be computed in multiple ways

- here form factor of the stress tensor multiplet in N=4 SYM

$$F = \langle g_1 g_2 | T(q) \rangle$$

cf electromagnetic
form factors in basic QFT



(simplicity: single scale problem)

$$p_1^2 = p_2^2 = 0$$

- arises in IR divergences: two internal/external momenta collinear or one momentum soft

- must cancel out in total cross-sections: imposes severe restrictions on observables (long story)

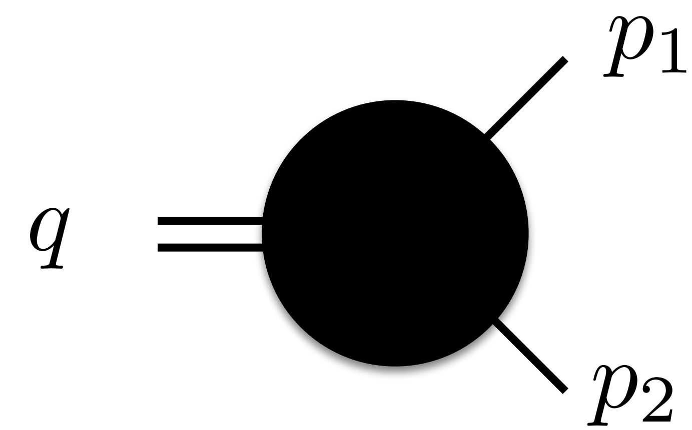
Sudakov form factor

$$F = \langle g_1 g_2 | T(q) \rangle$$

IR divergences 'exponentiate', roughly:

dim reg

$$A_l \propto e^{\frac{g_{\rm ym}^{2l}}{\epsilon^{2l}} h(g_{\rm ym}, N_c, \epsilon)} \tilde{A}$$



$$p_1^2 = p_2^2 = 0$$

involves universal functions, e.g $\gamma_{\rm cusp}$

- N=4 form factor factorises off a tree by SUSY,

$$F^{\mathcal{N}=4} = F^{(0)}\, \tilde{F}(g_{\rm ym}, N_c)$$

$\gamma_{\rm cusp}(g_{\rm ym}, N_c)$

$\simeq$ Form Factor

and exponentiates very easily

$$\mathrm{Log}[\tilde{F}] \propto \sum_l (-q^2)^{-l\epsilon} \frac{-g^{2l} \gamma_{\rm cusp}^{(l)}}{4(l\epsilon)^2} + \mathcal{O}(\epsilon^{-1})\,.$$

Sudakov form factor at four loops

Conjecture based on a variety of inputs on IR divergences:

non-planar correction to our cusp at four loops

- vanishes [Becher-Neubert, 09]
- probably [Ahrens-Neubert-Vernazza, 09]

"when in doubt, compute"

- integrand generation [RB-Kniehl-Tarasov-Yang, 12]
- IBP reduction [this talk, with caveats]
- (numerical) integration [this talk, partly]

Wise words

Feynman graphs generate high powers of irreducible numerators (will be out of reach)

→ need other method

"small problems at high loop orders are not small problems"

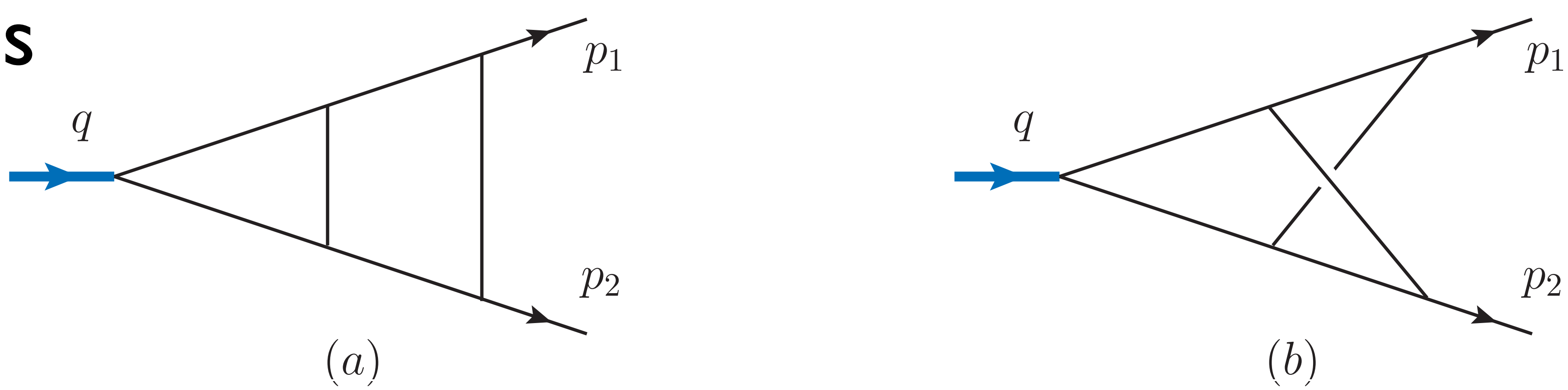
(≤ [Bern])

Integrand generation (N=4 case)

inspired by amplitude computation [Bern-et.al,12]:

- draw all trivalent graphs, dress with color & kinematics, relate numerators by color-kinematic duality
- feed in expectations about answer: UV divergences, absence of one-loop triangle graphs, symmetries
- check Ansatz using multicuts

e.g. 2pt @ 2 loops



- 2 'no-triangle' graphs
- no loop momenta in numerators allowed
- duality relates the 2 graphs

$$\mathcal{F}_2^{(2)} = s_{12}^2 \mathcal{F}^{(0)} \sum_{\sigma_2} \sum_{i=a}^{b} \frac{1}{S_i} C_i \, I_i \,,$$

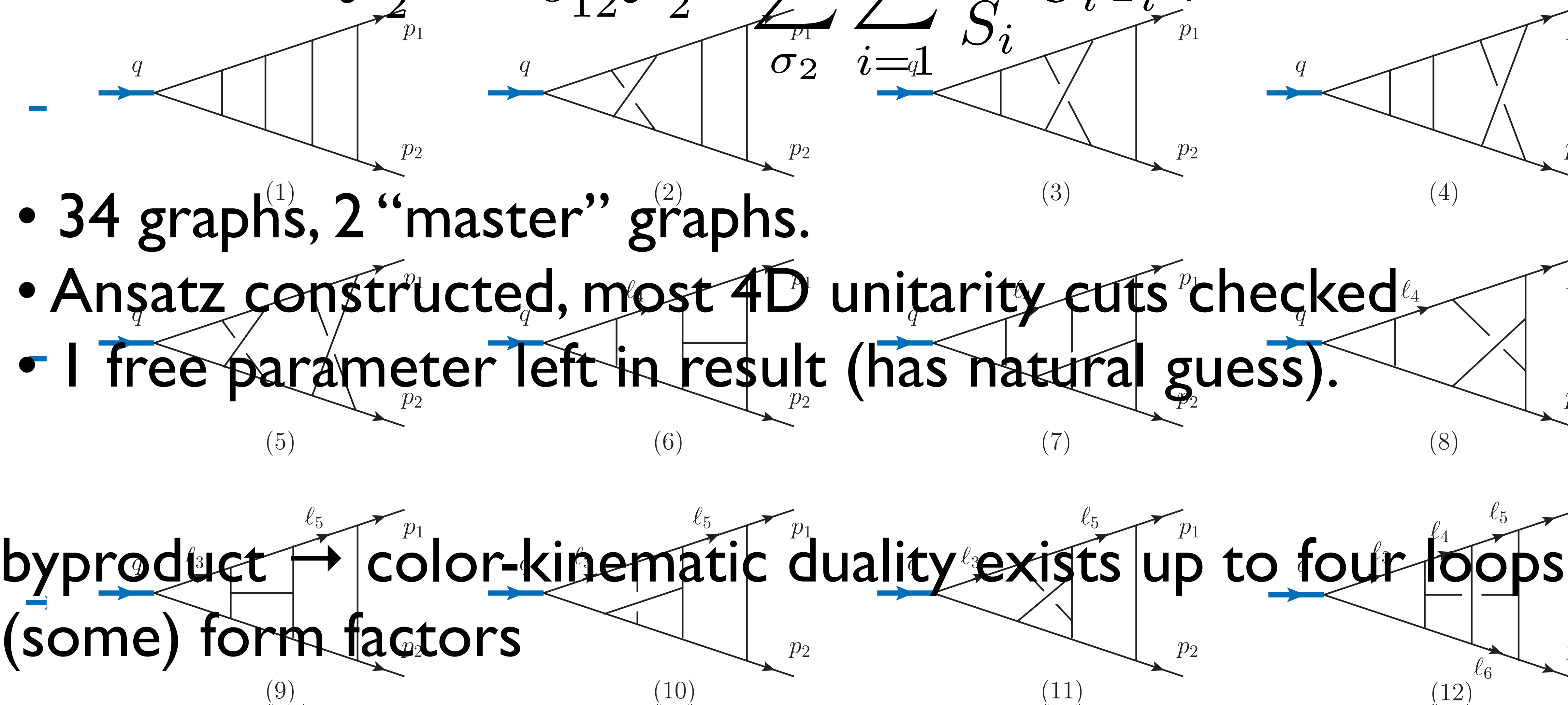
$$= N_c^2 \, \delta^{a_1 a_2} \, s_{12}^2 \, (4\, I_1 + I_2) \quad ✓$$

Integrand generation

[RB-Kniehl-Tarasov-Yang]

- checked 3 loop-2 point, 2 loop-3 point results (simple!)
- result for 4 loop-2 point:

$$\mathcal{F}_2^{(4)} = s_{12}^2 \mathcal{F}_2^{(0)} \sum_{\sigma_2} \sum_{i=1}^{34} \frac{1}{S_i} C_i \, I_i \, .$$



- 34 graphs, 2 "master" graphs.
- Ansatz constructed, most 4D unitarity cuts checked
- 1 free parameter left in result (has natural guess).

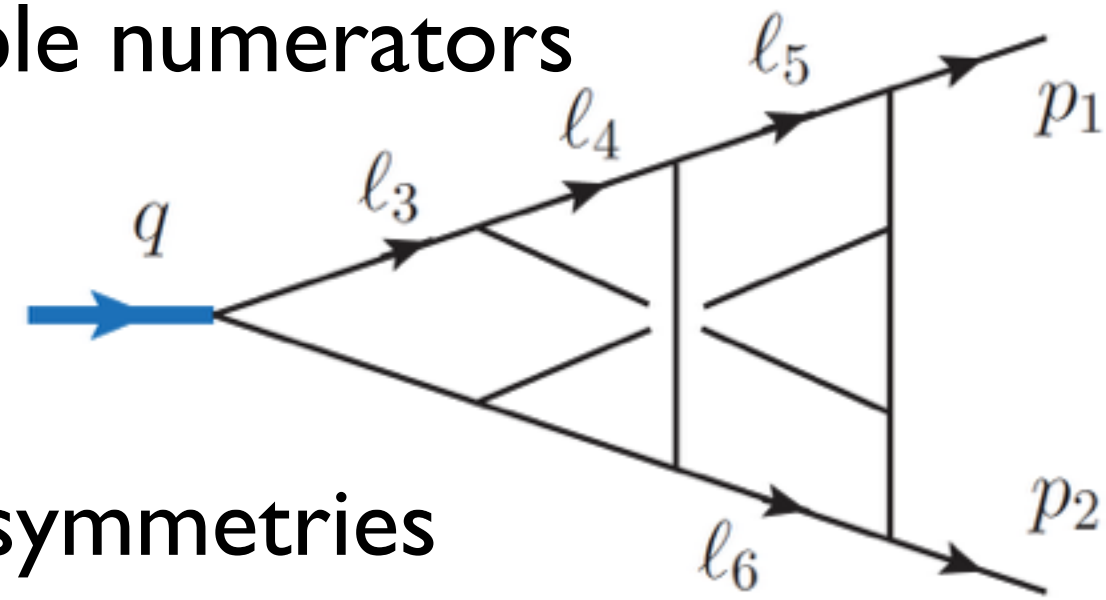
byproduct → color-kinematic duality exists up to four loops for (some) form factors

Integrand for N=4, published so far

Integral statistics after generation:

- 34 integrals, non-planar topologies rampant
- 13 have a non-planar color part
- 10 are purely non-planar color

- many up to quadratic in irreducible numerators
- topology 26: no internal boxes



(26)

- several have one or more graph symmetries

- generically, 18 independent propagators, 6 irreducible numerators / graph topology

IBP reduction

non-planar topology integrals with up-to quadratic numerators are hard to integrate → need simpler integrals

integrals obey many relations, e.g. IBPs: $\int d^D l_i \partial_{l_i^\mu} X = 0$

massive linear system → solve by mapping to matrix & row-reducing [Laporta, xx]

requires order on integrals

output: reduction + minimal basis of masters

here: for each integral pick 12 'propagators' to match topology → choose 6 'numerators' to make basis:

$$\forall x, y \in \{p_1, p_2, l_1, l_2, l_3, l_4\} \quad \exists\, \alpha_i \,|\, x \cdot y = \sum_i \alpha_i b_i$$

IBP reduction

IBPs implemented in many ways. Public:

- AIR [Anastasiou, Lazopoulos, 04],
- FIRE [Smirnov(s), 06,13,14], talk here at this workshop,
- Reduze [Studerus, 09], [Von Manteuffel-Studerus, 12]
- LiteRed [Lee, 12,13]

problems here: intermediate expression swell, extreme memory requirements, angry fellow users, crashes, etc, etc.

$\rightarrow$ none work out of the box for all integrals (try 26)

Reduze works after fixing disk access pile-up problem

- two choices of numerators (simplest vs most symmetric) tried, only simplest seems to work (?)

runtime: 4-5 months, regular interruptions, LARGE memory use

IBP reduction

Large memory use on single machine:

11031	boels	20	0	18.9g	18g	16m	R	64.2	1.9	6147:50	reduze
11024	boels	20	0	16.6g	16g	16m	R	53.4	1.6	5864:25	reduze
11032	boels	20	0	15.7g	15g	20m	R	63.2	1.6	6058:05	reduze
11093	boels	20	0	15.3g	15g	25m	R	99.3	1.5	5997:52	reduze
11038	boels	20	0	14.5g	14g	24m	R	100.0	1.4	6444:40	reduze
11044	boels	20	0	14.5g	14g	12m	R	100.0	1.4	5258:38	reduze
11087	boels	20	0	14.5g	14g	16m	R	98.3	1.4	5519:41	reduze
11035	boels	20	0	14.4g	14g	13m	R	100.0	1.4	5660:43	reduze
11091	boels	20	0	14.4g	14g	22m	R	75.7	1.4	6635:26	reduze
11036	boels	20	0	14.3g	14g	14m	R	62.3	1.4	6230:30	reduze
11085	boels	20	0	14.3g	14g	13m	R	69.5	1.4	6133:47	reduze
11030	boels	20	0	13.7g	13g	21m	R	93.7	1.4	5396:21	reduze
11041	boels	20	0	13.6g	13g	12m	R	100.0	1.3	6550:45	reduze
11027	boels	20	0	13.5g	13g	15m	R	60.3	1.3	5835:31	reduze
11088	boels	20	0	13.2g	13g	10m	R	100.0	1.3	5135:12	reduze
11025	boels	20	0	12.7g	12g	23m	R	67.8	1.3	5784:08	reduze
11028	boels	20	0	12.6g	12g	12m	R	51.4	1.2	6609:43	reduze
11042	boels	20	0	12.3g	12g	14m	R	51.4	1.2	5718:09	reduze
11090	boels	20	0	12.3g	12g	12m	R	100.0	1.2	6565:02	reduze
11095	boels	20	0	11.7g	11g	25m	R	91.4	1.2	5593:43	reduze
11043	boels	20	0	11.7g	11g	19m	R	65.5	1.2	5523:44	reduze
11039	boels	20	0	11.5g	11g	12m	R	100.0	1.1	6213:14	reduze
11033	boels	20	0	11.3g	11g	11m	R	65.2	1.1	5780:59	reduze
11029	boels	20	0	11.2g	10g	14m	R	99.6	1.1	5996:48	reduze
11086	boels	20	0	10.6g	10g	11m	R	94.7	1.0	5672:34	reduze
11023	boels	20	0	9.9g	9.7g	14m	R	79.0	1.0	6489:06	reduze
11136	boels	20	0	9798m	9.4g	27m	R	91.8	0.9	6417:35	reduze
11026	boels	20	0	9308m	9.0g	27m	R	90.4	0.9	6421:29	reduze
11037	boels	20	0	9022m	8.7g	19m	R	59.0	0.9	5713:30	reduze
11040	boels	20	0	8876m	8.5g	14m	R	57.3	0.9	6139:33	reduze
11092	boels	20	0	8431m	8.1g	20m	S	100.0	0.8	6483:24	reduze
11089	boels	20	0	7900m	7.6g	21m	R	69.5	0.8	6322:00	reduze
11116	boels	20	0	7044m	6.7g	26m	R	100.0	0.7	6417:08	reduze
11034	boels	20	0	7042m	6.7g	20m	R	100.0	0.7	6022:42	reduze

IBP reduction: Reduze

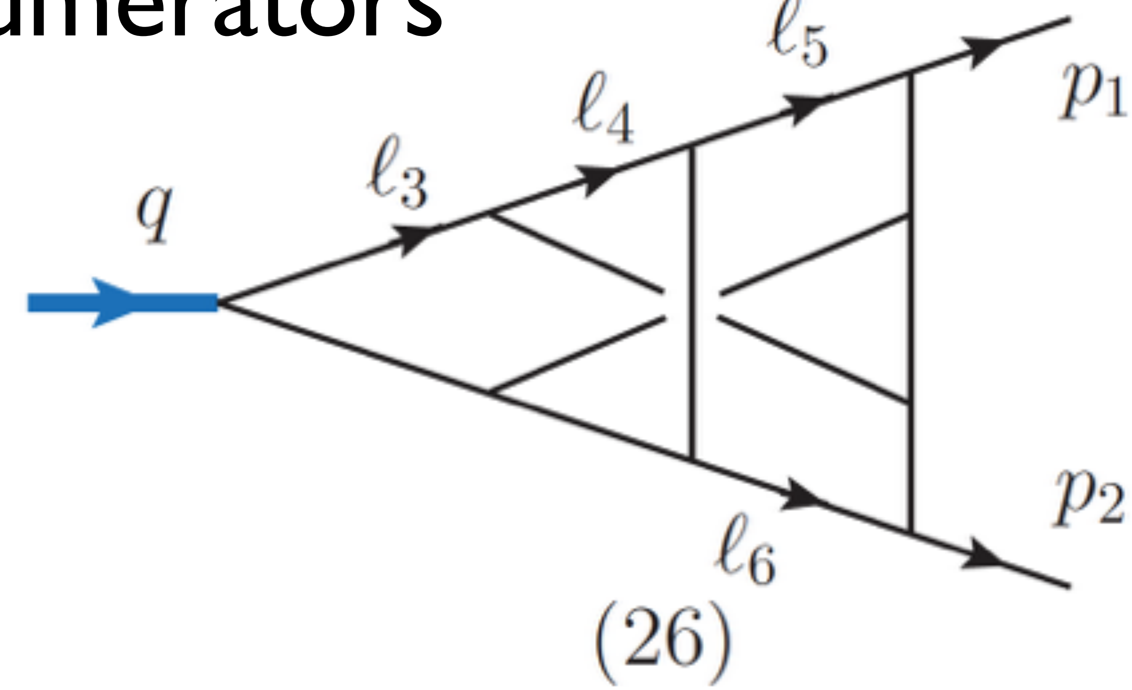
Reduze

+++ works on the problem at hand
+ scales well with number of topologies
+ enables large scale parallel computing

- requires much disk space, memory
- parallel increase saturates ~10-100 processes

— does not scale well with numerator or propagator power

IBP reduction: choice of numerators



take e.g. topology 26:

has graph symmetry of order 4

(graph has one more symmetry exchanging gluon and q)

numerators can be chosen to either simple or symmetric:

simple	symmetric
$(l_3 - l_5)^2\,,$	$(l_3 + 2l_5 - 3p_1)^2\,,$
$(l_3 - l_6)^2\,,$	$(-l_3 + 2l_6 + p_2)^2\,,$
$(l_5 - l_6)^2\,,$	$(l_3 - 2l_6 - p_1 + 2\,p_2)^2\,,$
$(l_4 - p_1)^2\,,$	$(l_3 + 2l_5 - p_2)^2\,,$
$(l_4 - p_2)^2\,,$	$(l_3 - l_4 + l_5 - p_2)^2\,,$
$(l_5 - p_2)^2\,.$	$(l_4 - l_6 - p_1 + p_2)^2\,.$

Wise words

"small problems at high loop orders are not small problems"

($\leq$ [Bern])

generating symmetric numerators is not easy,
choices involved, fews days of work

IBP reduction: output

Reduze solves finite ranges of identities: choice up to 2 numerator powers, up to 12 denominator powers
(extension to 13 under way, beyond unrealistic)

one unreduced master detected (file size) → obtained from symmetry

Table 1: Master integral statistics of obtained IBP reduction

(a) planar form factor

# props	$s=0$	s=1	$s=2$
12	8	6	0
11	18	2	0
10	43	9	0
9	49	1	0
8	51	4	1
7	25	0	0
6	8	0	0
5	0	0	0
sum	203	22	1

(b) non-planar form factor

# props	$s=0$	s=1	$s=2$
12	10	10	1
11	13	3	0
10	34	10	0
9	29	1	0
8	32	3	1
7	13	0	0
6	7	0	0
5	1	0	0
sum	139	27	2

hardest

Wise words

"small problems at high loop orders are not small problems"

(≤ [Bern])

how do you know the answer makes sense?

→ crosscheck

Basis check from MINT

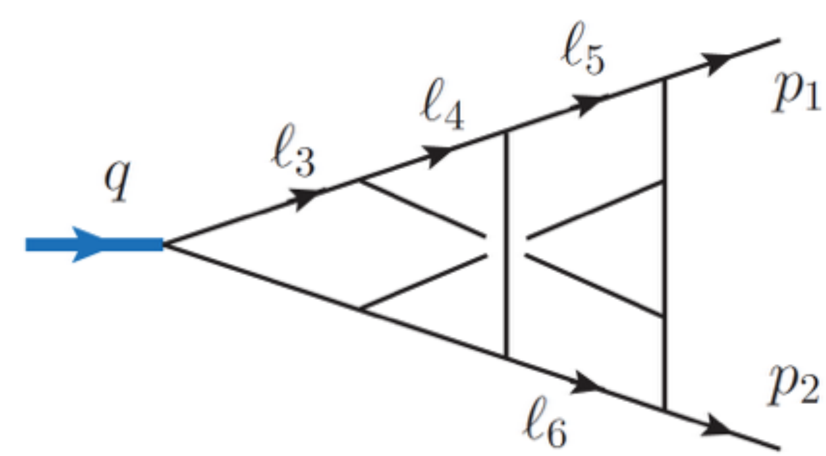
observation [Lee, Pomeransky, 13]: "number of master integrals in given sector from algebraic geometry"

- determine physical subsectors, e.g. with LiteRed
- compute $G = F + U$ via Feynman parameter integral for each
- look for roots of: $I = \left\langle \frac{\partial G}{\partial \alpha_1}, \ldots, \frac{\partial G}{\partial \alpha_m}, \alpha_0 G - 1 \right\rangle,$
- hard → compute Gröbner basis
  - Mathematica
  - Macaulay 2
  - Singular
- further processing for hard cases as in [Lee, Pomeransky, 13]
- number of masters allows a choice of basis (typically corner)
- obtained a complete basis for all topologies (caveat)

Cross checks & integration

- MINT favors doubled up propagators, Reduze numerators
- checked all single basis integrals agree between MINT and Reduze, beyond numbers close
- hardest integral topology seems integral 26 involving quadratic numerator

choose numerator $(l3 \cdot (p_1 - p_2))^2$



(26)

$$
\begin{aligned}
=&(-0.032986 \pm 2.16391 \cdot 10^{-7})\,\epsilon^{-8} + (0.0694456 \pm 1.00572 \cdot 10^{-5})\,\epsilon^{-7} \\
&+ (1.3506 \pm 0.000193163)\,\epsilon^{-6} + (-2.68804 \pm 0.00316693)\,\epsilon^{-5} \\
&+ (-6.23707 \pm 0.0401289)\,\epsilon^{-4} + (12.6763 \pm 2.0782)\,\epsilon^{-3} \\
&+ (1234.49 \pm 32.9661)\,\epsilon^{-2} + \mathcal{O}(\epsilon^{-1})\,.
\end{aligned}
$$

(FIESTA)

Integration status

- Mellin-Barnes for non-planar at four loops open problem
- AMBRE & MB & Cuba still useful for some integrals

- sector_decomposition, secdec 2.x insufficient here

- FIESTA can do most integrals for planar form factor
  - up to three 12 propagator integrals
  - 2x integral 25, 1 x integral 30

- likely that precision is a problem - observed order 0.1

(Reduze+FIESTA give the three loop cusp in ~ 2 days up to percent level)

Outlook

progress reported toward four loop form factors (any theory)

- basis of masters
- cross-checks in place

most definitely the wrong way to compute!

- better basis? Von Manteuffel talk
- solve IBPs with four dots to open more possibilities

Your Idea
Here?