Generalised unitarity for dimensionally regulated amplitudes within FDF

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Based on work with:
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and
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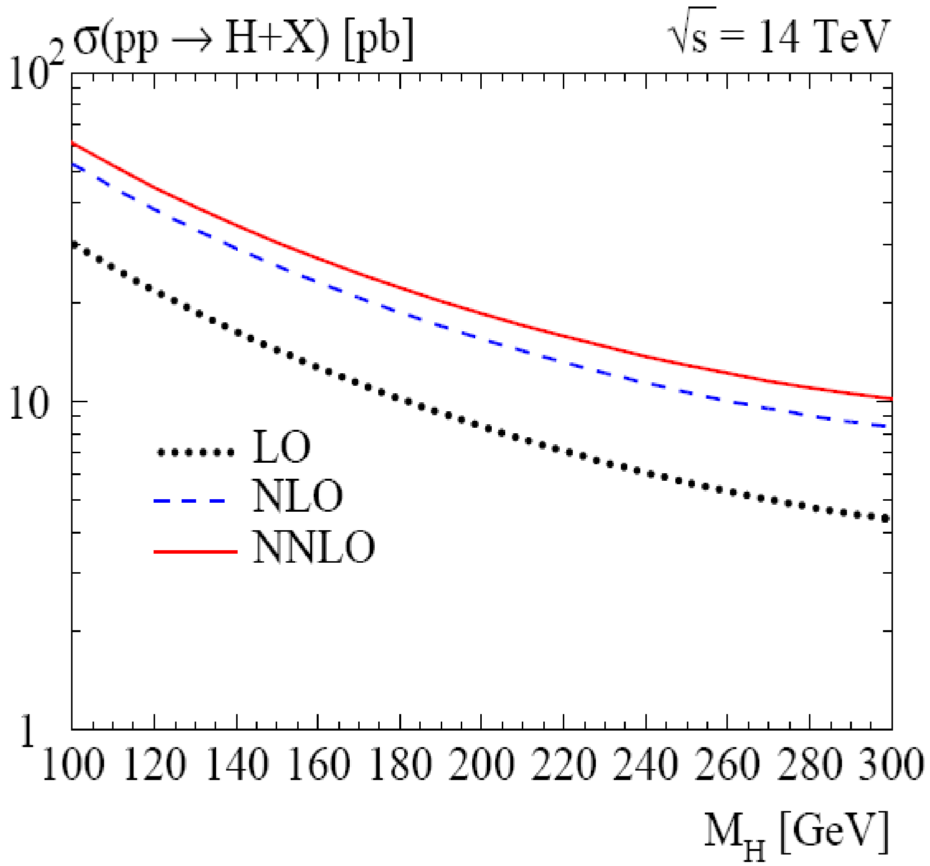
Introduction

- Scattering amplitudes are necessary to test our theoretical models by comparing their predictions against the experiments.

Harlander, Kilgore (2002), Anastasiou, Melnikov (2002)



- Tree-level (LO) predictions are qualitative due to the poor convergence of the truncated expansion at strong coupling.

$$\alpha_S\,(100\mathrm{GeV}) \sim 0.12$$

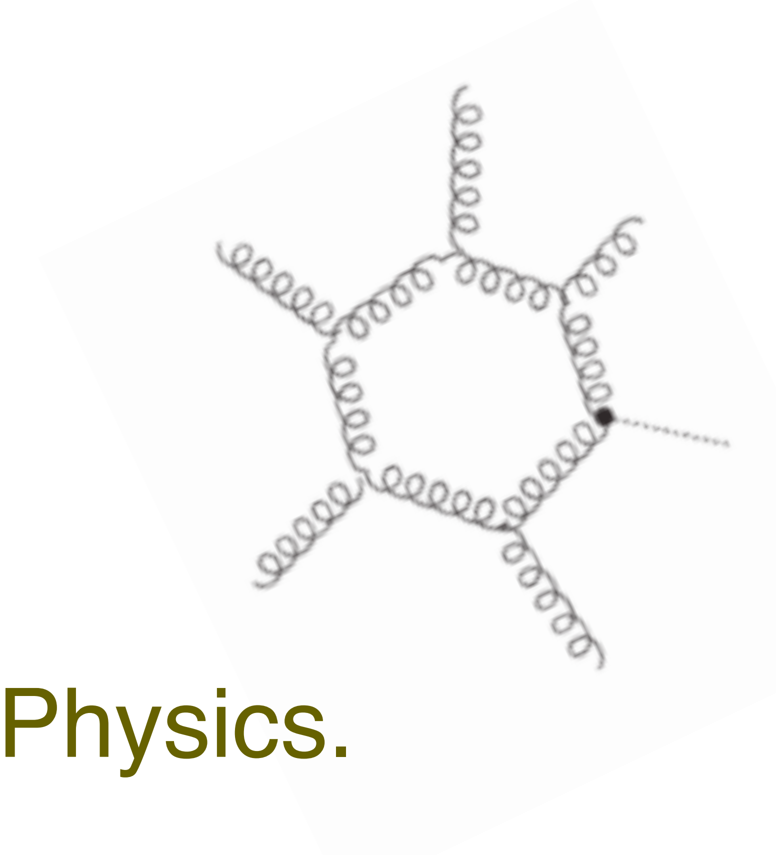
- K factors

$$K = \frac{\mathrm{NLO}}{\mathrm{LO}} \sim 30\% \div 80\%$$

- Feynman diagrams, based on the Lagrangian, are not optimised for these processes.

- On-shell methods are based on amplitudes and take full advantage of the analyticity of the S-matrix.

Motivation



- Simplify the calculations in High-Energy Physics.
- Discover hidden properties of Quantum Field Theories
- Towards NNLO is the Next Frontier.

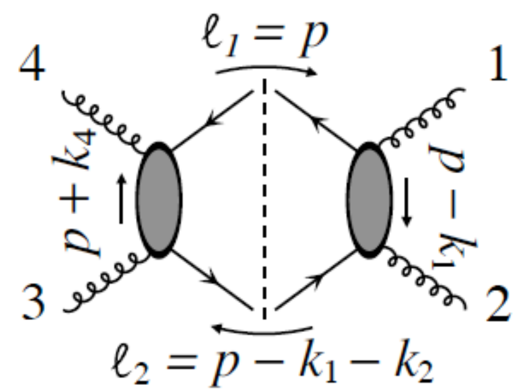
Outline

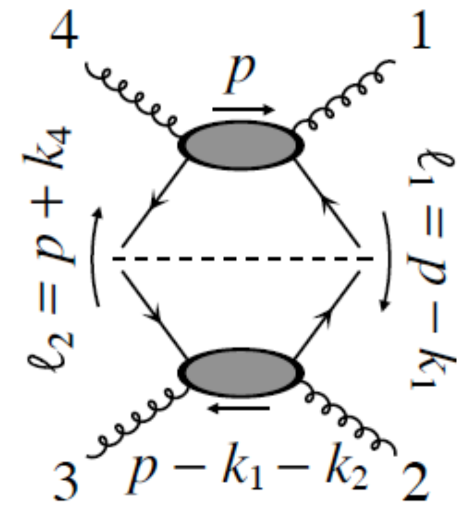
- Analytic one-loop amplitudes
  — *d* dimensional generalised unitarity
  — Four dimensional formulation of dimensional regularisation
- Automation of analytic one-loop amplitudes
- Results
- Off-shell colour kinematics duality
- **Conclusions**

The Method

Standard Unitarity in 4D

[Bern, Dixon, Dunbar, Kosower (1994)]





Glue together the two amplitudes and uplift the integral with

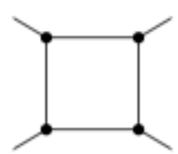
$$2\pi\delta^{(+)}\left(p^2 - m^2\right) \rightarrow \frac{i}{p^2 - m^2 - i\epsilon}$$

Generalised Unitarity in 4D

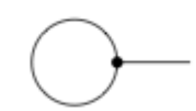
[Bern, Dixon, Kosower (1998); Britto, Cachazo, Feng (2004)]

$$\mathcal{A}^{(L)} = \sum_i c_i \mathcal{I}_i^{(L)} \longrightarrow$$ Known basis of L-loop scalar integrals

For L=1, [Passarino - Veltman (1979)]

$$A_n^{(1),D=4}(\{p_i\}) = \sum_{K_4} C^{[0]}_{4;K4}$$

$$+ \sum_{K_3} C^{[0]}_{3;K3}$$

$$+ \sum_{K_2} C^{[0]}_{2;K2}$$

$$+ \sum_{K_1} C^{[0]}_{1;K1}$$


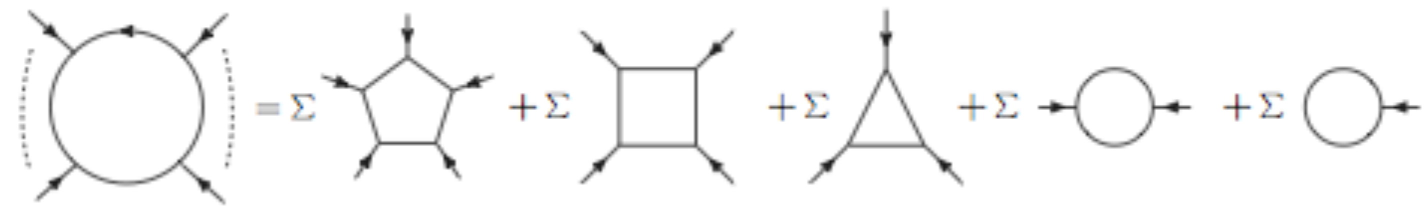
Scalar Master Integrals: Made of polylogarithmic functions

- If an amplitude is determined by its branch cuts, it is said to be cut-constructible.
- All one-loop amplitudes are cut-constructible in dimensional regularisation.

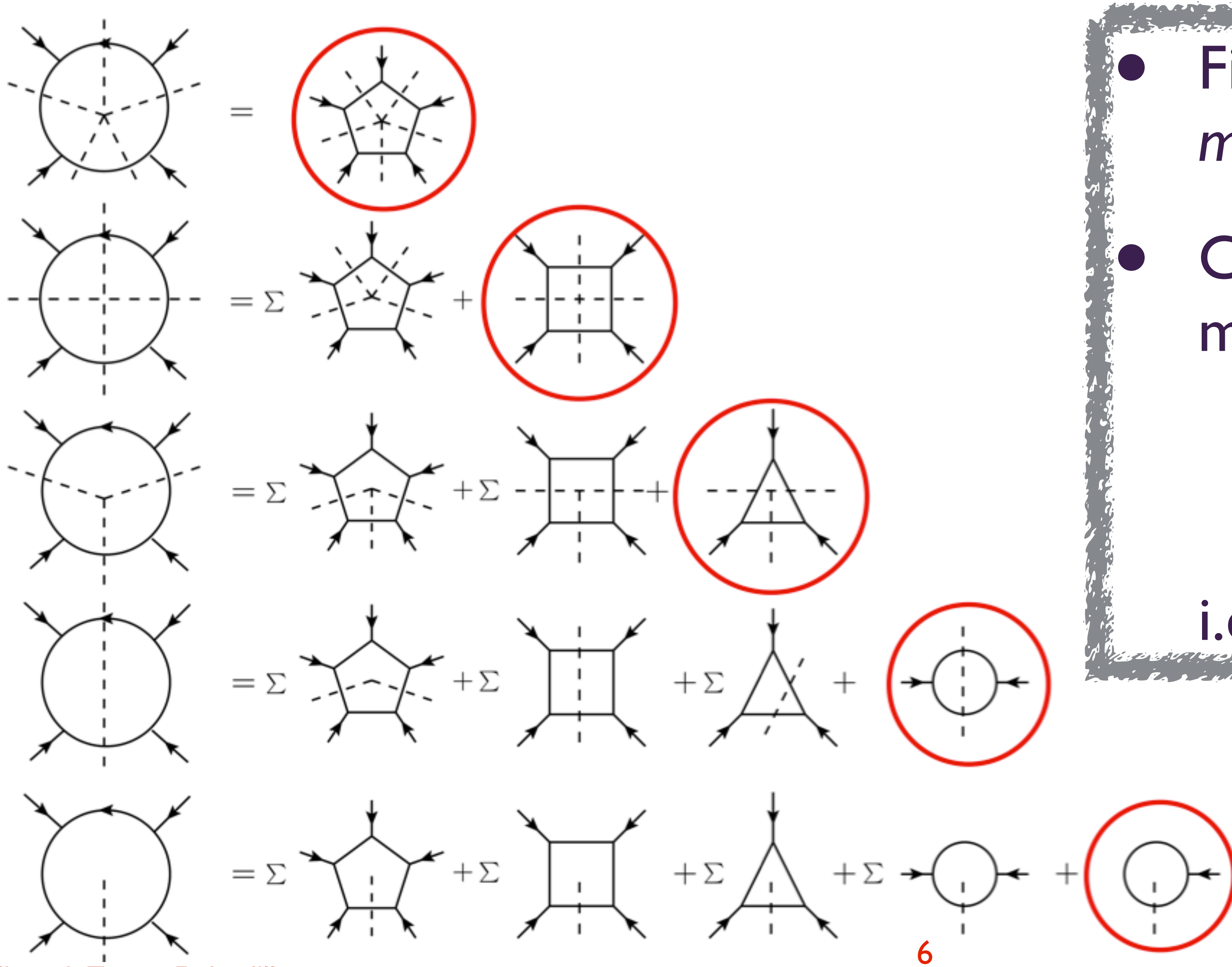
Generalised Unitarity: isolate the leading discontinuity

[Ossola, Papadopulos, Pittau (2007)]
[Ellis, Giele, Kunszt, Melnikov (2009)]

Integrand decomposition:



See G. Ossola's Talk



- Fit m-point residues on m-ple cuts
- Cutting a loop propagator means

$$\frac{1}{D_i} \to 2\pi\delta^{(+)}\left(D_i\right)$$

i.e. putting it on-shell

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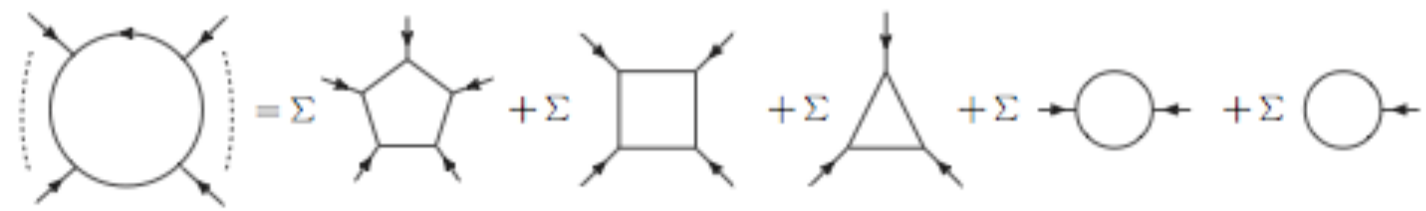

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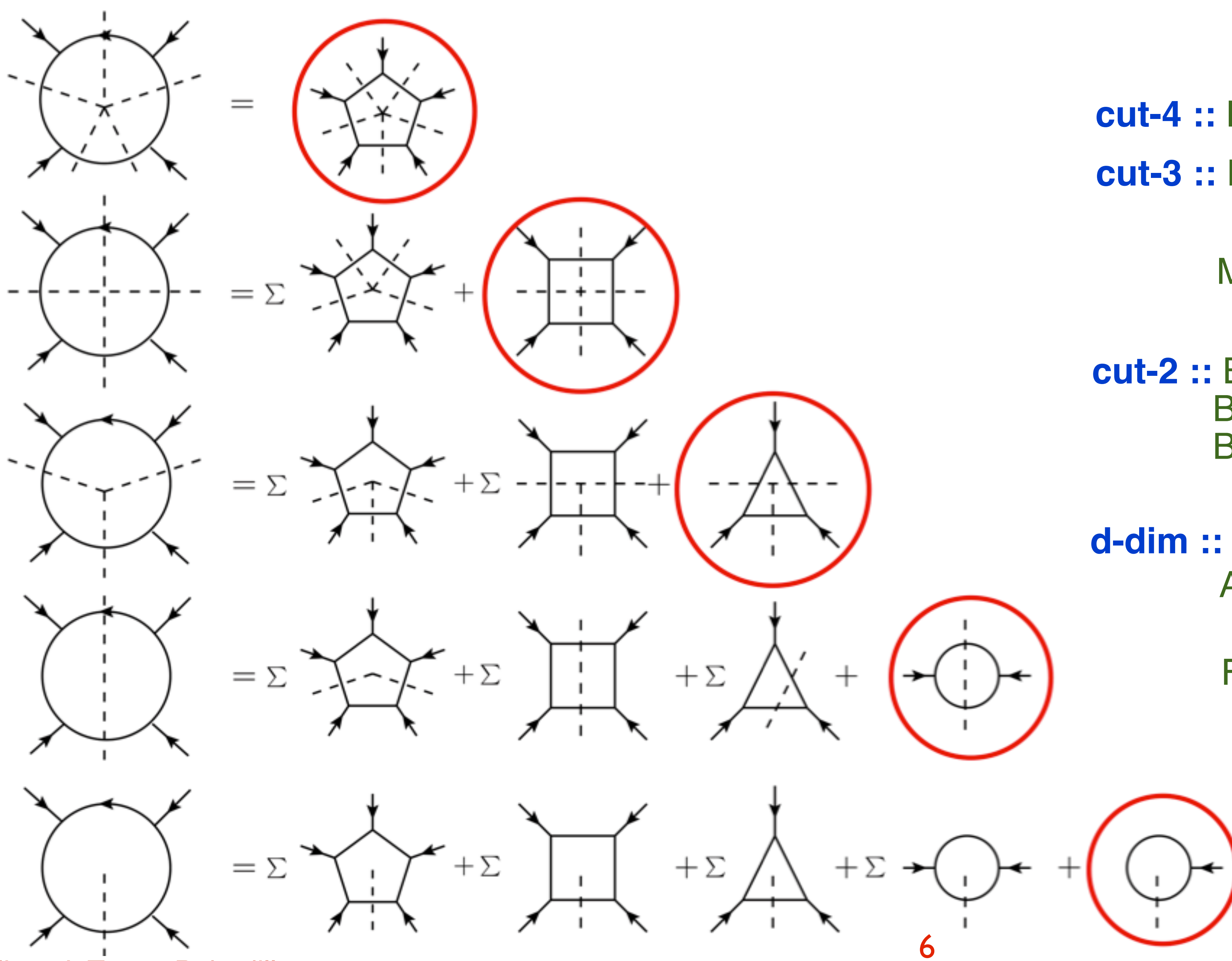

cut-4 :: Britto Cachazo Feng

cut-3 :: Forde
Bjerrum-Bohr, Dunbar, Ita, Perkins
Mastrolia

cut-2 :: Bern, Dixon, Dunbar, Kosower.
Britto, Buchbinder, Cachazo, Feng.
Britto, Feng, Mastrolia.

d-dim :: Bern, Morgan.
Anastasiou, Britto, Feng, Kunszt, Mastrolia
Ellis Giele Kunszt Melnikov Badger
Fazio, Mastrolia, Mirabella, **W.T.**

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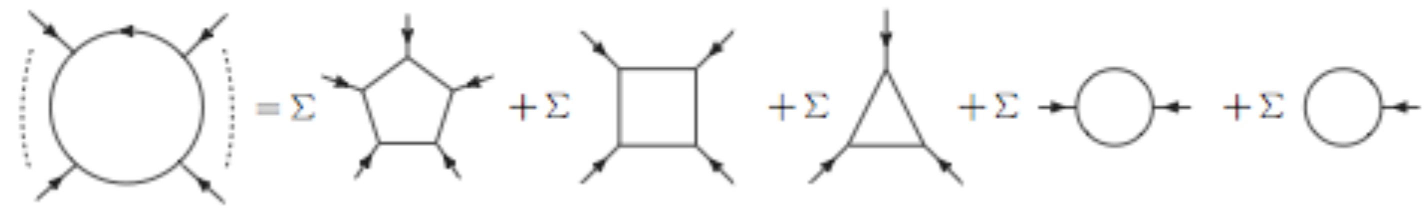

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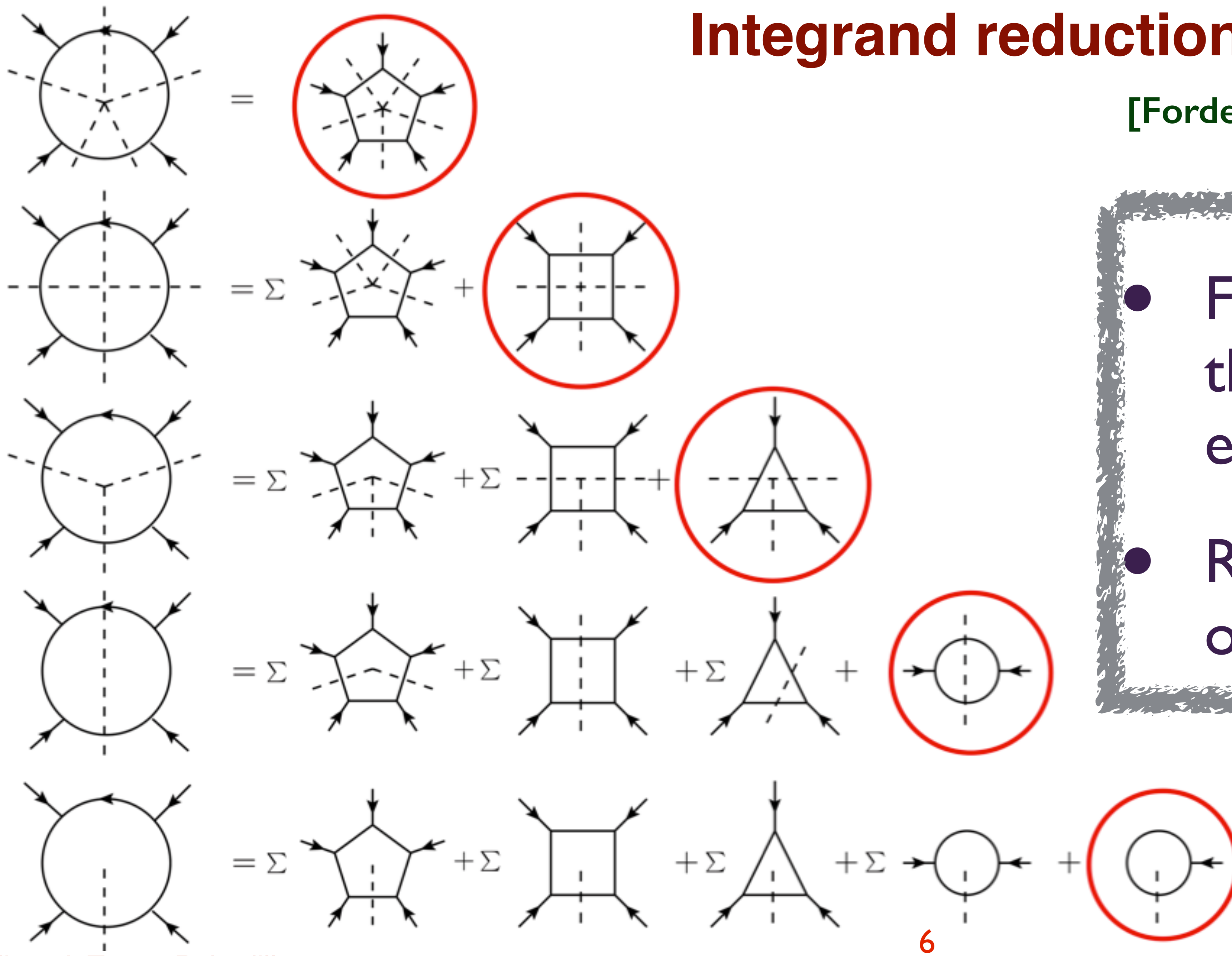


Integrand reduction via Laurent expansion

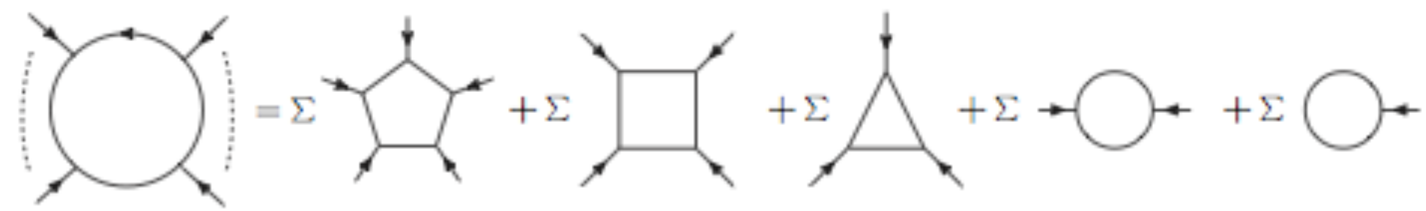
[Forde (2007); Kilgore (2008), Badger (2009)]
[Mastrolia, Mirabella, Peraro (2012)]

- Fit residues by taking their asymptotic expansions of the cuts
- Requires the computation of fewer coefficients

Generalised Unitarity: isolate the leading discontinuity

[Ossola, Papadopulos, Pittau (2007)]
[Ellis, Giele, Kunszt, Melnikov (2009)]

Integrand decomposition:



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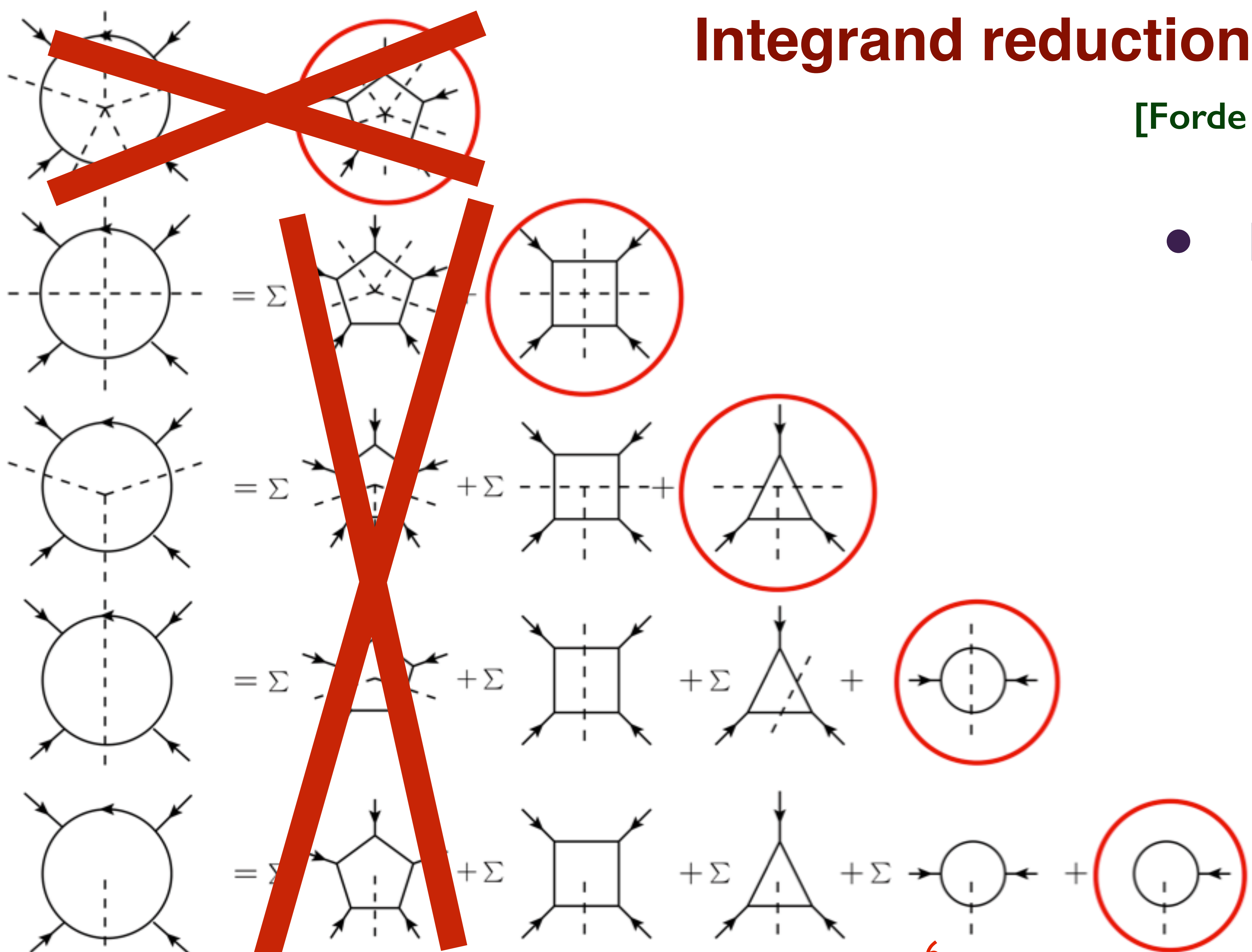

Integrand reduction via Laurent expansion

[Forde (2007); Kilgore (2008), Badger (2009)]
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- Pentagons not needed

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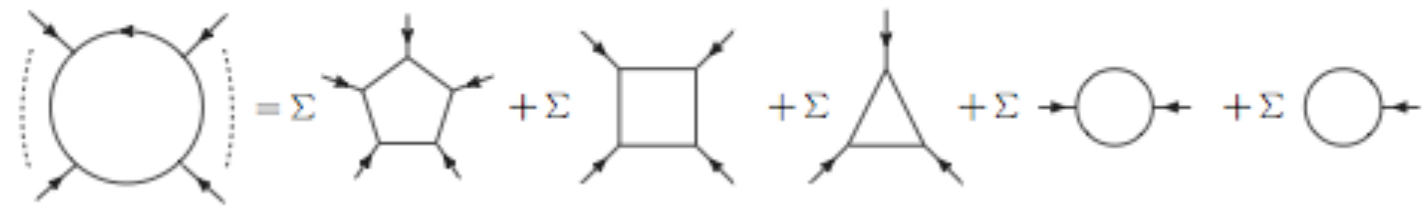
Integrand decomposition:



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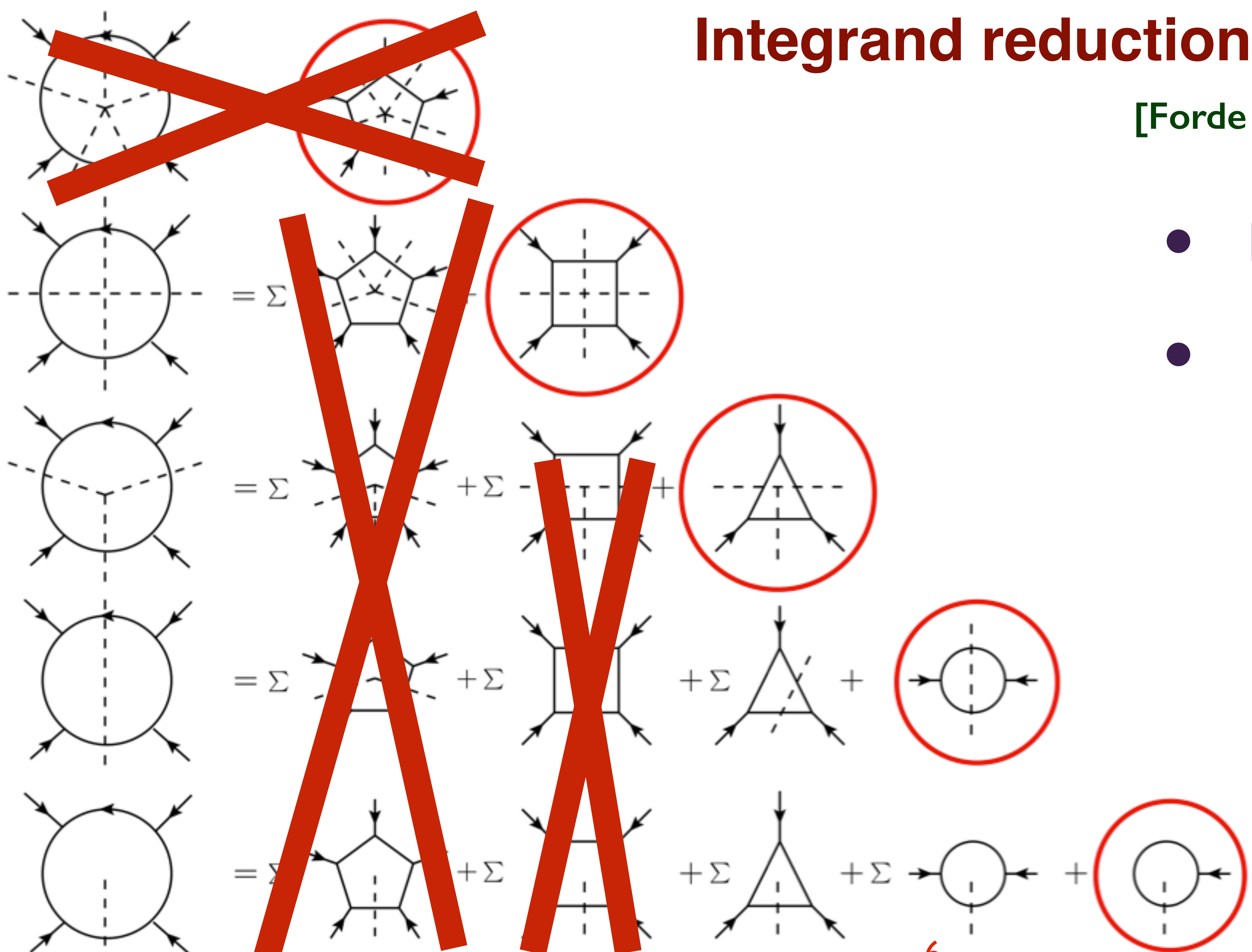

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- Pentagons not needed
- Boxes never subtracted out

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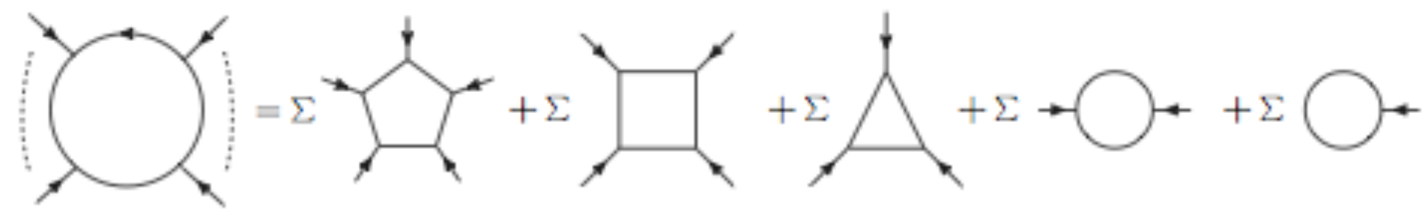

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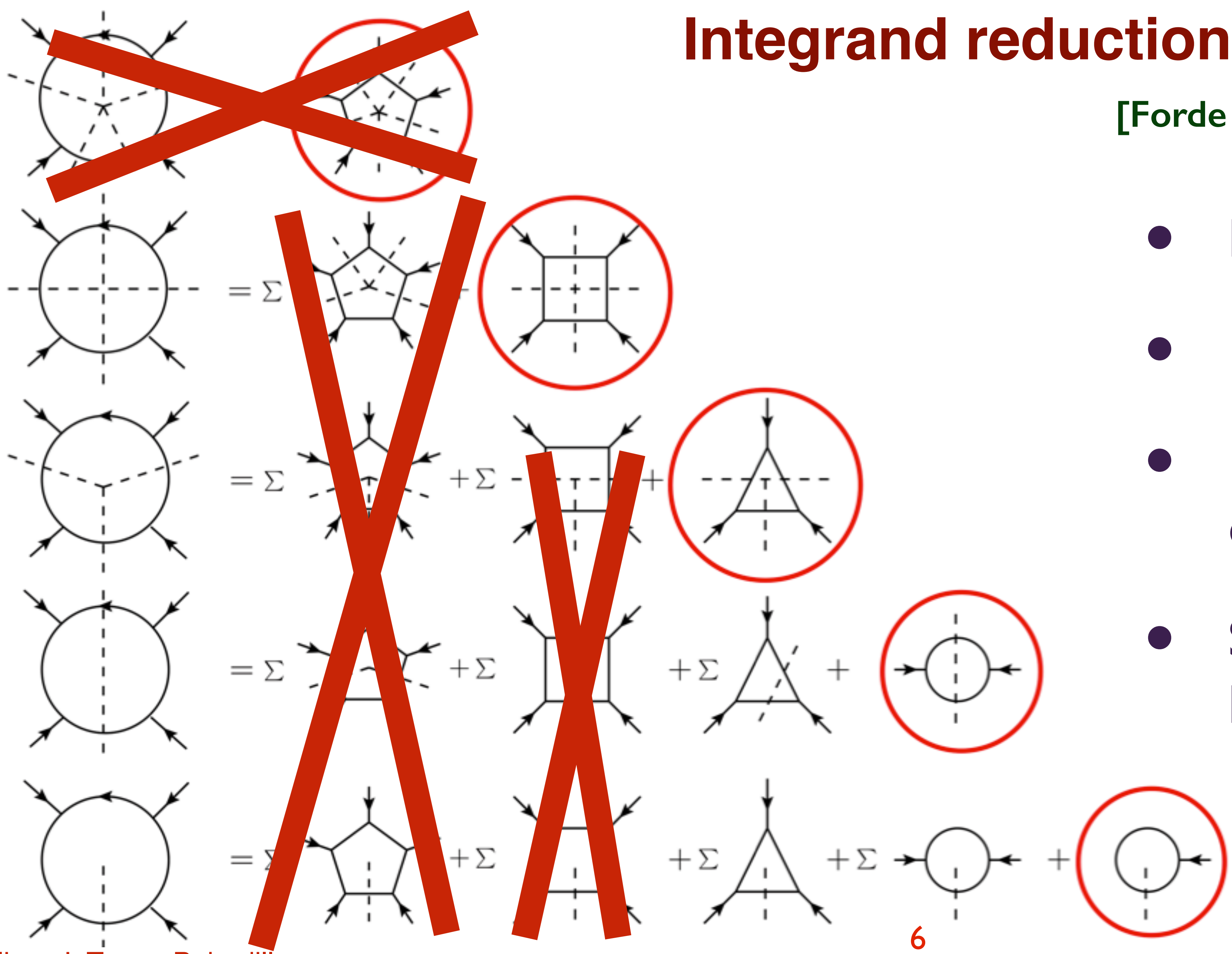


Integrand reduction via Laurent expansion

[Forde (2007); Kilgore (2008), Badger (2009)]
[Mastrolia, Mirabella, Peraro (2012)]

- Pentagons not needed
- Boxes never subtracted out
- Diagonal system of equations
- Subtraction at coefficient level

D-dimensional cuts

In $D = 4 - 2\epsilon$ we can do the decomposition

$$\bar{\ell}^\nu = \ell^\nu + \tilde{\ell}^\nu$$

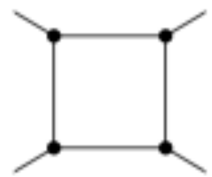
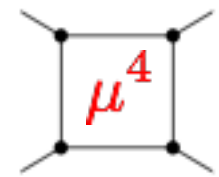
$D = 4$ $\qquad$ $D = -2\epsilon$

The on-shell condition

$$\bar{\ell}^2 = \ell^2 - \mu^2 = 0 \longrightarrow \ell^2 = \mu^2$$

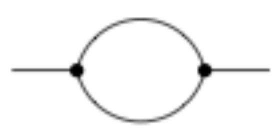
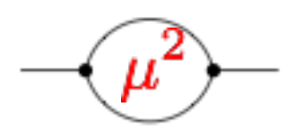
Mass term

- Any massless one-loop becomes

$$A_n^{(1),D=4-2\epsilon}(\{p_i\}) = \sum_{K_4} C_{4;K4}^{[0]}$$

$$+ \sum_{K_4} C_{4;K4}^{[4]}$$


$$+ \sum_{K_3} C_{3;K3}^{[0]}$$

$$+ \sum_{K_3} C_{3;K3}^{[2]}$$


$$+ \sum_{K_2} C_{2;K2}^{[0]}$$

$$+ \sum_{K_2} C_{2;K2}^{[2]}$$


$$+ \sum_{K_1} C_{1;K1}^{[0]}$$


[Bern,Dixon,Dunbar,Kosower (1997)]
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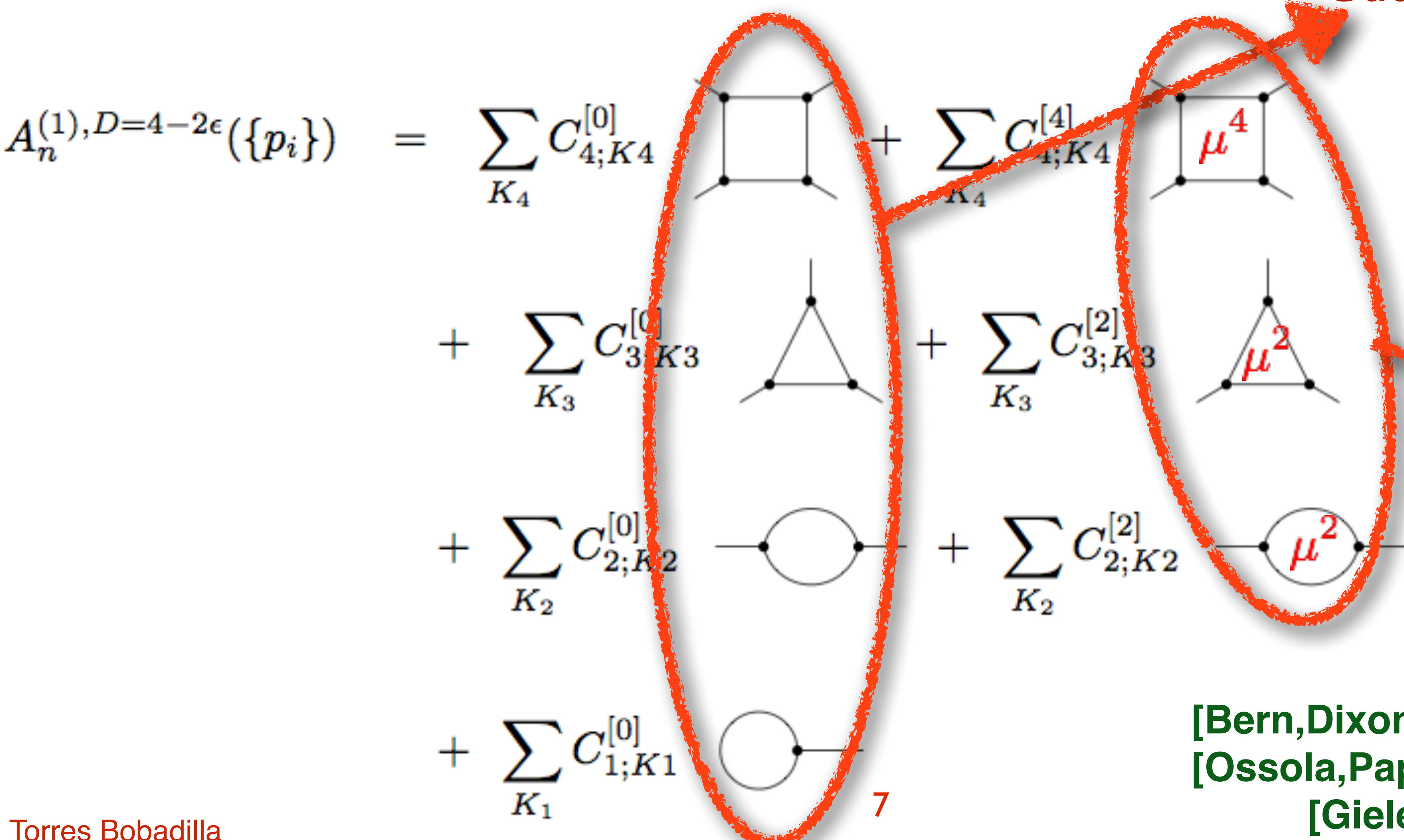
Cut-constructible part



Rational part

[Bern,Dixon,Dunbar,Kosower (1997)]
[Ossola,Papadopoulos,Pittau (2006)]
[Giele,Kunszt,Melnikov (2008)]

How to compute these coefficients?

To compute amplitudes at1-loop and understand how to treat cuts in D-dimensions there are existing approaches

A: Separated computation of cut-constructible and rational terms

A1: Computing the rational term separately (using non gauge invariant terms)

— **R1 and R2 separation** [Ossola, Papadopoulos, Pittau(2008); Pittau, Draggiotis, Garzelli (2009)]
— **Supersymmetric decomposition** [Bern, Dixon, Kosower]

B: D-dimensional unitarity offers the determination of all pieces together

B1: 6-dimensional spinor-helicity formalism [Cheung and O'Connell(2009); Davies (2012)]

— **New rules for spinor products**
— **No automatic generator exists**

B2: Gamma algebra in extended dimension [Ellis,Giele,Kunszt,Melnikov (2008)]

— **The explicit representation of the polarisation states is avoid**
— **Gamma algebra has to be extended everywhere.**
— **Automatic generator has to be modified**

B3: Don't leave 4 dimensions! [Fazio, Mastrolia, Mirabella, WT (2014)]

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B: D-dimensional unitarity offers the determination of all pieces together

Four Dimensional Formulation of Dimensional Regularisation (FDF)

B3: Don't leave 4 dimensions! [Fazio, Mastrolia, Mirabella, WT (2014)]

— Explicit 4D representation of polarisation and spinors
— 4D representation of D-reg loop propagators
— 4D Feynman rules + (-2ε)-Selection Rules
— Easy to implement in existing generators

FDH: 4D helicity scheme

[Bern and Kosower (1992)]

The d-dimensional metric tensor can be split as $\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}$

d-dimensional

4-dimensional

-2ε-dimensional

Where

$$\tilde{g}^{\mu\nu} g_{\mu\nu} = 0, \qquad \tilde{g}^{\mu}_{\mu} = -2\epsilon \xrightarrow[d\to 4]{} 0, \qquad g^{\mu}_{\mu} = 4 \qquad \tilde{q}^2 = \tilde{g}^{\mu\nu} \bar{q}_\mu \bar{q}_\nu = -\mu^2$$

Projections of the vectors q and $\tilde{q}$.

$$\tilde{q}^{\mu} g_{\mu\nu} = \tilde{g}^{\mu\sigma} \bar{q}_\sigma g_{\mu\nu} = 0$$

As well for the gamma matrices

$$[\tilde{\gamma}^\alpha, \gamma^5] = 0, \qquad \{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2\,\tilde{g}^{\alpha\beta}, \qquad \{\tilde{\gamma}^\alpha, \gamma^\mu\} = 0.$$

Can we implement it with 4D-object only?

In 4-dimensions, one can infer: $\tilde{\gamma} \sim \gamma^5$

And the Clifford algebra $\tilde{\gamma}^\mu \tilde{\gamma}_\mu \xrightarrow[d\to 4]{} 0 \quad \text{while} \quad \gamma^5\gamma^5 = 1$

Excludes any four-dimensional representation of the —2ε-subspace

—2ε-subspace → —2ε-Selection Rules (—2ε)-SRs

[Fazio, Mastrolia, Mirabella, WT (2014)]

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-2ε-Selection Rules

The Clifford algebra conditions are satisfied by imposing

$$\tilde{g}^{\alpha\beta} \to G^{AB}, \qquad \tilde{\ell}^\alpha \to i\,\mu\,Q^A, \qquad \tilde{\gamma}^\alpha \to \gamma^5\,\Gamma^A.$$

A,B := —2ε-dimensional vectorial indices traded for (—2ε)-SRs

$$\begin{aligned} G^{AB}G^{BC} &= G^{AC}, & G^{AA} &= 0, & G^{AB} &= G^{BA}, \\ \Gamma^A G^{AB} &= \Gamma^B, & \Gamma^A\Gamma^A &= 0, & Q^A G^A &= 1, \\ Q^A G^{AB} &= Q^B, & Q^A Q^A &= 1. \end{aligned}$$

[Fazio, Mastrolia, Mirabella, WT (2014)]

Gluon propagator

[Fazio, Mastrolia, Mirabella, WT (2014)]

The helicity sum of the transverse polarisation vector is

$$\sum_{i=1}^{d-2} \varepsilon^{\mu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right)\varepsilon^{*\nu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right) = -\bar{g}^{\mu\nu} + \frac{\bar{\ell}^{\mu}\,\bar{\eta}^{\nu} + \bar{\ell}^{\nu}\,\bar{\eta}^{\mu}}{\bar{\ell}\cdot\bar{\eta}},$$

We choose $\bar{\eta}^{\mu} = \bar{l}^{\mu} - \tilde{l}^{\mu}$ (gauge invariance in d-dimensions)

In d = 4-2ε:

$$\sum_{i=1}^{d-2} \varepsilon^{\mu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right)\varepsilon^{*\nu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right) = \underbrace{\left(-g^{\mu\nu} + \frac{\ell^{\mu}\ell^{\nu}}{\mu^2}\right)}_{d\,=\,4} - \underbrace{\left(\tilde{g}^{\mu\nu} + \frac{\tilde{\ell}^{\mu}\tilde{\ell}^{\nu}}{\mu^2}\right)}_{d\,=\,-2\varepsilon}.$$

massive gluon

$$\left(-g^{\mu\nu} + \frac{l^{\mu}l^{\nu}}{\mu^2}\right) = \sum_{\lambda=\pm,0} \varepsilon^{\mu}_{\lambda}\left(l\right)\varepsilon^{*\nu}_{\lambda}\left(l\right)$$

-2ε-scalar

$$\left(\tilde{g}^{\mu\nu} + \frac{\tilde{l}^{\mu}\tilde{l}^{\nu}}{\mu^2}\right) \longrightarrow \hat{G}^{AB} = G^{AB} - Q^{A}Q^{B}$$

Fermion propagator

[Fazio, Mastrolia, Mirabella, WT (2014)]

The spinors of a d-dimensional fermion have to fulfil the completeness relation

$$\sum_{\lambda=1}^{2^{(d-2)/2}} u_{\lambda,(d)}\left(\bar{l}\right)\bar{u}_{\lambda,(d)}\left(\bar{l}\right) = \bar{\not{l}} + m$$

$$\sum_{\lambda=1}^{2^{(d-2)/2}} v_{\lambda,(d)}\left(\bar{l}\right)\bar{v}_{\lambda,(d)}\left(\bar{l}\right) = \bar{\not{l}} - m$$

The FDF allows us to express these relations as

$$\sum_{\lambda=\pm} u_{\lambda}\left(l\right)\bar{u}_{\lambda}\left(l\right) = \not{l} + i\mu\gamma^{5} + m$$

$$\sum_{\lambda=\pm} v_{\lambda}\left(l\right)\bar{v}_{\lambda}\left(l\right) = \not{l} + i\mu\gamma^{5} - m$$

Polarisation vectors and Spinors

[Fazio, Mastrolia, Mirabella, WT (2014)]

Spinors

Therefore, we can generalise the Dirac Equation

$$\left(\not{\ell} + i\mu\gamma^5 + m\right) u_\lambda(\ell) = 0\,, \quad \ell^2 = m^2 + \mu^2\,, \quad \ell = \ell^\flat + \frac{m^2+\mu^2}{2\,\ell\cdot q_\ell} q_\ell\,, \quad (\ell^\flat)^2 = (q_\ell)^2 = 0\,.$$

with the solutions

$$u_+(\ell) = \left|\ell^\flat\right\rangle + \frac{(m - i\mu)}{\left[\ell^\flat\, q_\ell\right]} \left|q_\ell\right]\,, \qquad u_-(\ell) = \left|\ell^\flat\right] + \frac{(m + i\mu)}{\left\langle \ell^\flat\, q_\ell\right\rangle} \left|q_\ell\right\rangle\,,$$

$$\bar{u}_+(\ell) = \left[\ell^\flat\right| + \frac{(m + i\mu)}{\left\langle q_\ell\, \ell^\flat\right\rangle} \left\langle q_\ell\right|\,, \qquad \bar{u}_-(\ell) = \left\langle\ell^\flat\right| + \frac{(m - i\mu)}{\left[q_\ell\, \ell^\flat\right]} \left[q_\ell\right|\,.$$

Polarisation Vectors

Analogous to the spinors we can build polarisation vectors for the internal lines

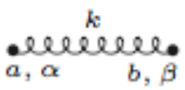
$$\varepsilon^\mu_+(\ell) = -\frac{\left[\ell^\flat\right|\gamma^\mu\left|\hat{q}_\ell\right\rangle}{\sqrt{2}\mu}\,, \qquad \varepsilon^\mu_-(\ell) = -\frac{\left\langle\ell^\flat\right|\gamma^\mu\left|\hat{q}_\ell\right]}{\sqrt{2}\mu}\,, \qquad \varepsilon^\mu_0(\ell) = \frac{\ell^{\flat\mu} - \hat{q}^\mu_\ell}{\mu}\,.$$

which fulfil the well-known relations for massive vector bosons

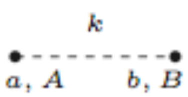
$$\hat{q}_\ell \equiv \frac{m^2+\mu^2}{2\,\ell\cdot q_\ell} q_\ell\,.$$

Feynman Rules in FDF

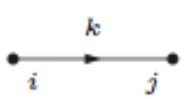
[Fazio, Mastrolia, Mirabella, WT (2014)]



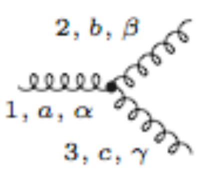
$$= -i\,\delta^{ab}\,\frac{1}{k^2-\mu^2+i0}\left[g^{\alpha\beta}-\frac{k^\alpha k^\beta}{\mu^2}\right] \quad \text{(gluon)},$$



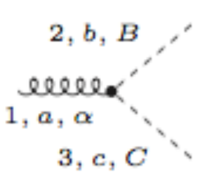
$$= -i\,\delta^{ab}\,\frac{G^{AB}}{k^2-\mu^2+i0}\,, \quad \text{(scalar)},$$



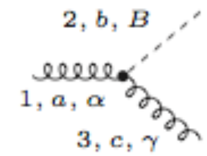
$$= i\,\delta^{ij}\,\frac{\not{k}+i\mu\gamma^5+m}{k^2-m^2-\mu^2+i0}\,, \quad \text{(fermion)},$$



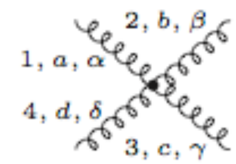
$$= -g\,f^{abc}\left[(k_1-k_2)^\gamma g^{\alpha\beta} + (k_2-k_3)^\alpha g^{\beta\gamma} + (k_3-k_1)^\beta g^{\gamma\alpha}\right],$$



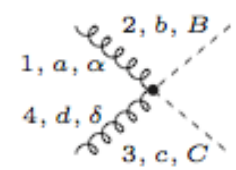
$$= -g\,f^{abc}\,(k_2-k_3)^\alpha\,G^{BC}\,,$$



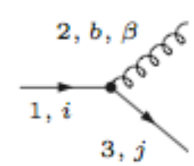
$$= \mp g\,f^{abc}\,(i\mu)\,g^{\gamma\alpha}\,Q^B \qquad (\tilde{k}_1 = 0, \quad \tilde{k}_3 = \pm\tilde{\ell}),$$



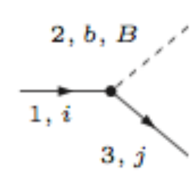
$$= -ig^2\Big[f^{xad}\,f^{xbc}\,(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\gamma}g^{\beta\delta}) + f^{xac}\,f^{xbd}\,(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\delta}g^{\beta\gamma}) + f^{xab}\,f^{xdc}\,(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta})\Big],$$



$$= 2ig^2\,g^{\alpha\delta}\,\left(f^{xab}\,f^{xcd}+f^{xac}\,f^{xbd}\right)\,G^{BC}\,,$$



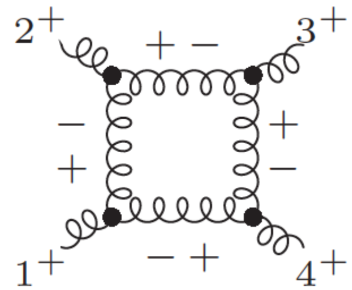
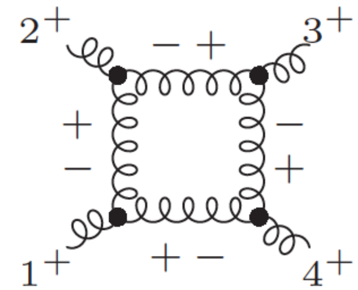
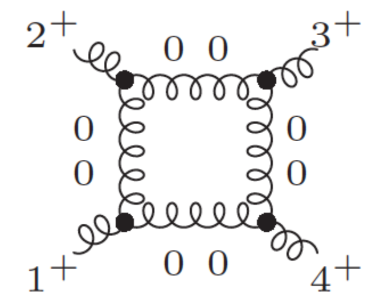
$$= -ig\,\left(t^b\right)_{ji}\,\gamma^\beta\,,$$

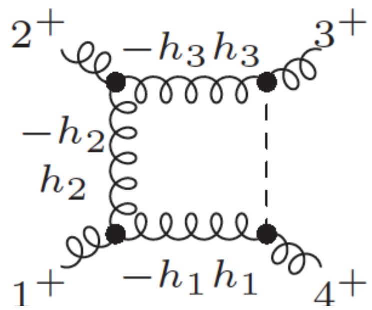


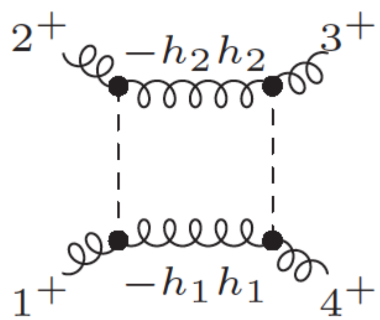
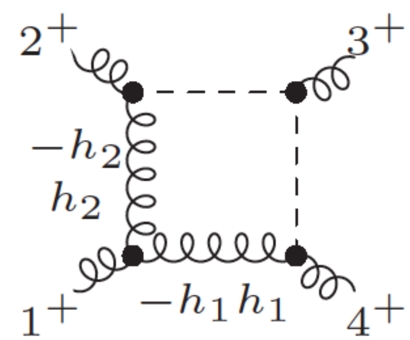
$$= -ig\,\left(t^b\right)_{ji}\,\gamma^5\,\Gamma^B\,.$$

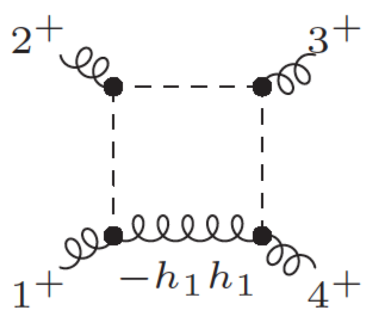
The simplest example: the gggg amplitude

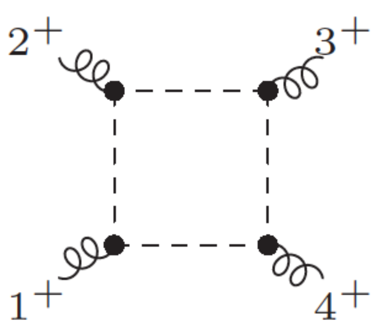
[Fazio, Mastrolia, Mirabella, WT (2014)]

$$C^{[0]}_{1|2|3|4} =$$  $+$  $+$ 

$$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$$  $+$ c.p. ,

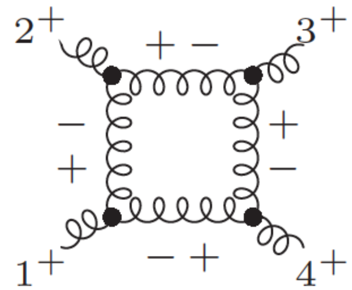
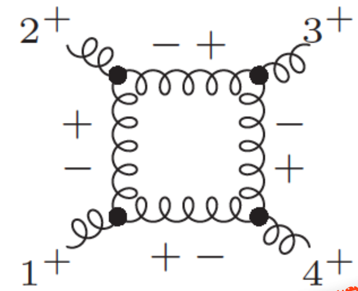
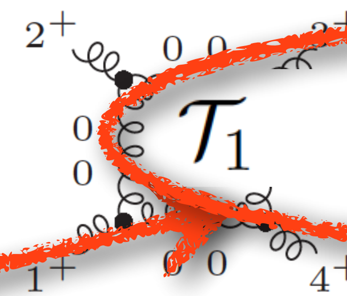
$$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$$  $+ \mathcal{T}_2$  $+$ c.p.

$$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$$  $+$ c.p.

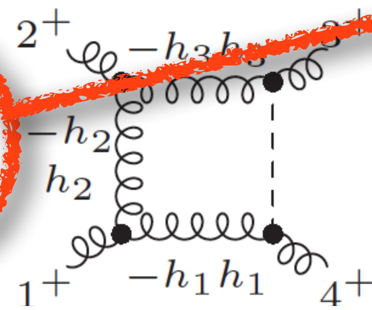
$$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$$ 

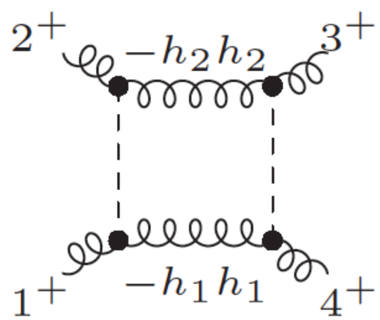
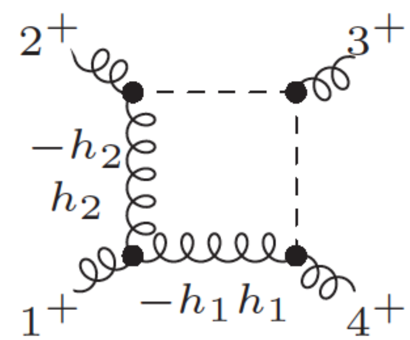
The simplest example: the gggg amplitude

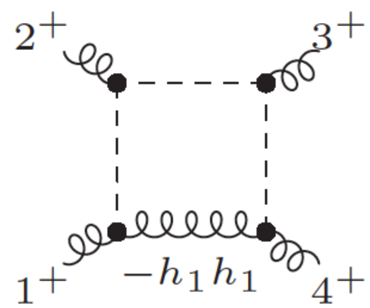
[Fazio, Mastrolia, Mirabella, WT (2014)]

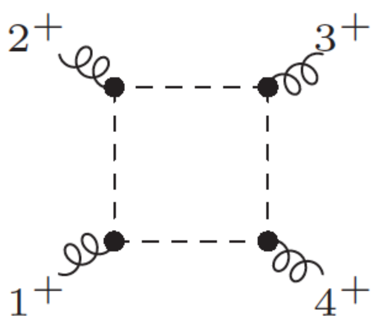
$$C^{[0]}_{1|2|3|4} =$$  $$+$$  $$+$$ 

$$\mathcal{T}_1 = Q^A \hat{G}^{AB} Q^B = 0$$

$$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$$  $$+ \text{c.p.}\,,$$

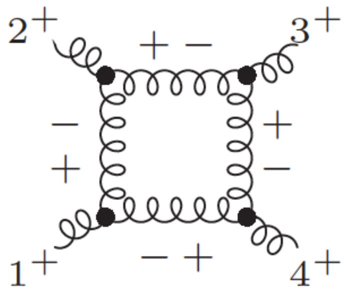
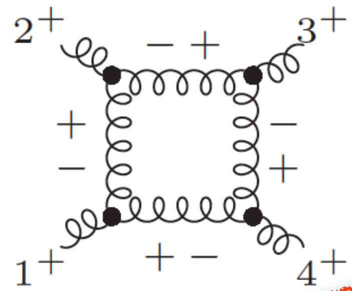
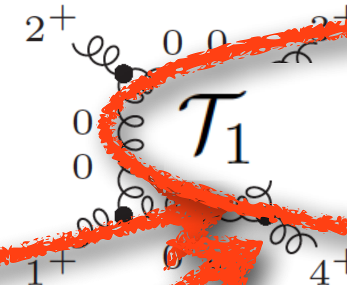
$$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$$  $$+ \mathcal{T}_2$$  $$+ \text{c.p.}$$

$$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$$  $$+ \text{c.p.}$$

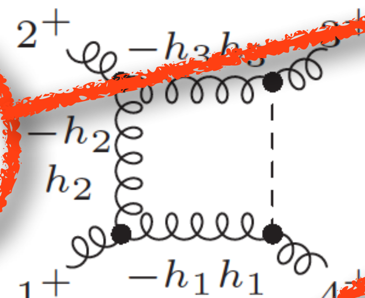
$$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$$ 

The simplest example: the gggg amplitude

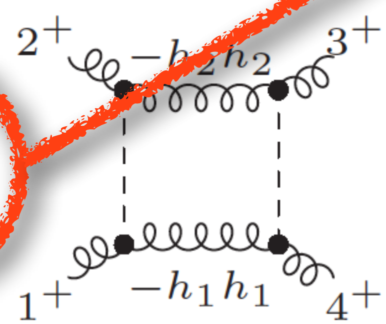
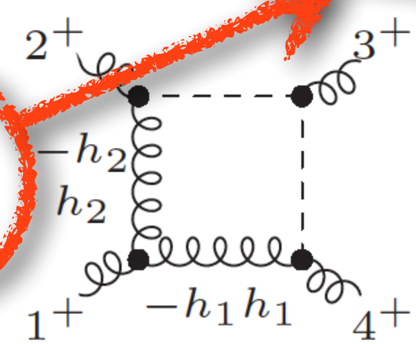
[Fazio, Mastrolia, Mirabella, WT (2014)]

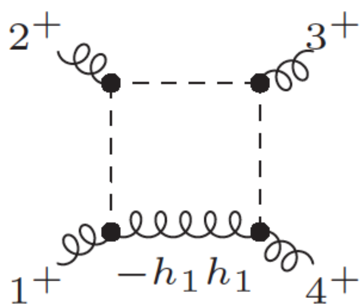
$$C^{[0]}_{1|2|3|4} =$$  $+$  $+$ 

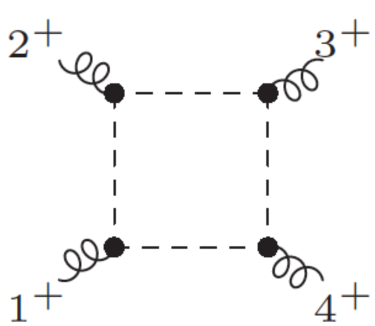
$$\mathcal{T}_1 = Q^A \hat{G}^{AB} Q^B = 0$$

$$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$$  $+\,\text{c.p.},$

$$\mathcal{T}_2 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} Q^D = 0$$

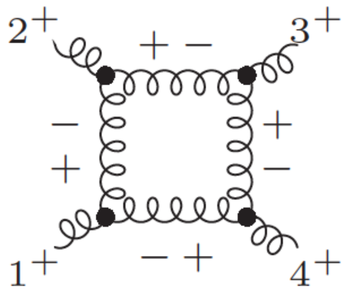
$$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$$  $+\ \mathcal{T}_2$  $+\,\text{c.p.}$

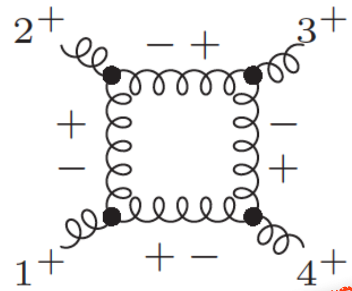
$$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$$  $+\,\text{c.p.}$

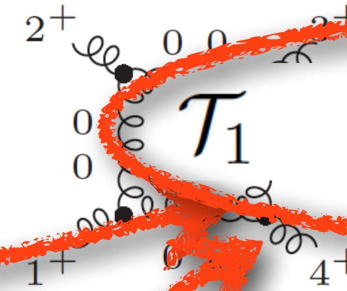
$$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$$ 

The simplest example: the gggg amplitude

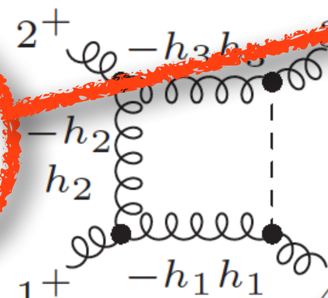
[Fazio, Mastrolia, Mirabella, WT (2014)]

$C^{[0]}_{1|2|3|4} =$ 

$+$ 

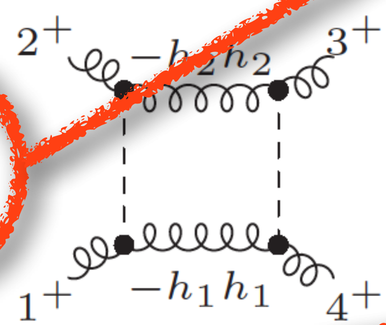
$+$ 

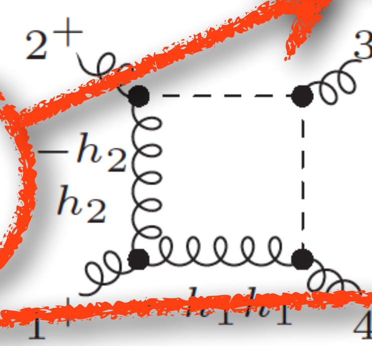
$$\mathcal{T}_1 = Q^A \hat{G}^{AB} Q^B = 0$$

$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$ 

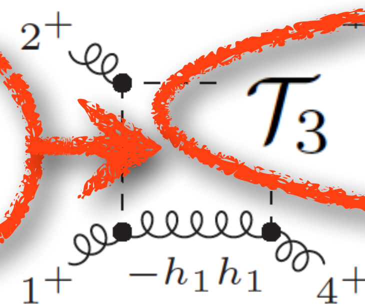
$+$ c.p.,

$$\mathcal{T}_2 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} Q^D = 0$$

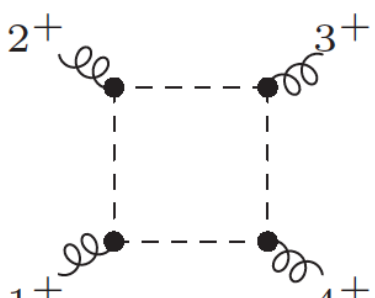
$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$ 

$+ \mathcal{T}_2$ 

$+$ c.p.

$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$ 

$$\mathcal{T}_3 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} G^{DE} \hat{G}^{EF} Q^F = 0$$

$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$ 

The simplest example: the gggg amplitude

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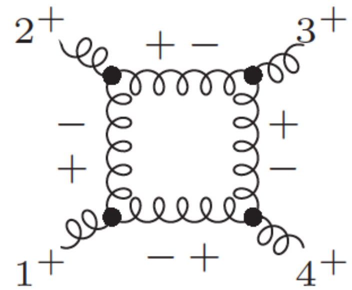
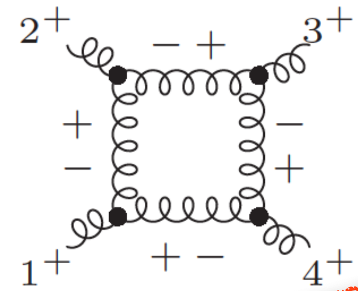
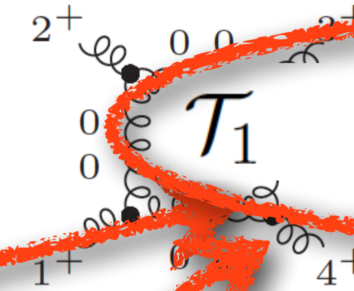


$+$

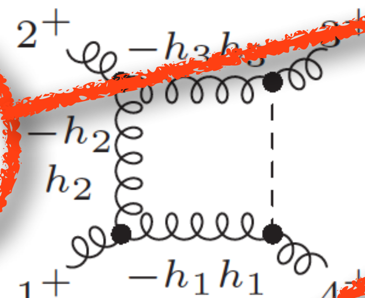


$+$



$$\mathcal{T}_1 = Q^A \hat{G}^{AB} Q^B = 0$$

$$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$$



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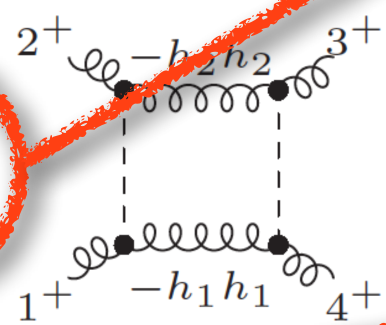
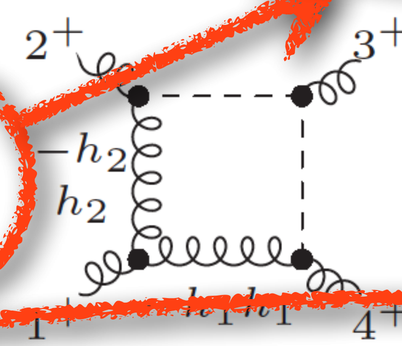
$$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$$



$+ \mathcal{T}_2$



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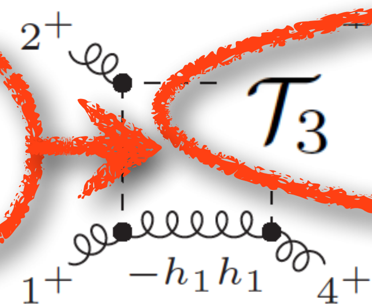
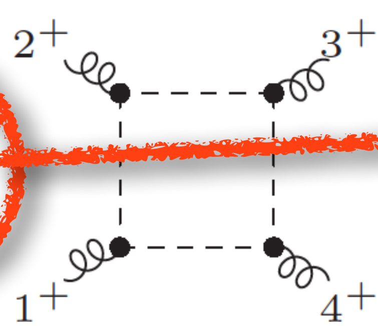
$$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$$



$$\mathcal{T}_3 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} G^{DE} \hat{G}^{EF} Q^F = 0$$

$$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$$



$$\mathcal{T}_4 = \mathrm{tr}\left(G\hat{G}G\hat{G}G\hat{G}G\hat{G}\right) = -1$$

The simplest example: the gggg amplitude

[Fazio, Mastrolia, Mirabella, WT (2014)]

$$A_4^{1-\text{loop}}\left(1_g^+, 2_g^+, 3_g^+, 4_g^+\right)$$

Contributions come only from the coefficients:

(Gluon loop) $\quad c^{[0]}_{1|2|3|4;\,0} = 0\,, \qquad c^{[0]}_{1|2|3|4;\,4} = 3i\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}$

(-2ε-Scalar loop) $\quad c^{[4]}_{1|2|3|4;\,0} = 0\,, \qquad c^{[4]}_{1|2|3|4;\,4} = -i\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}$

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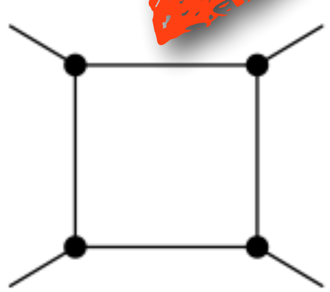
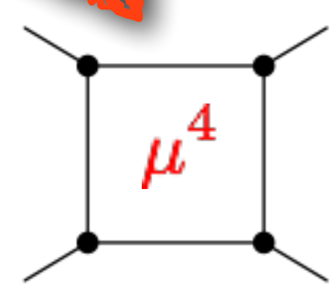
Recall

$$\sum_{K_4} C^{[0]}_{4;K4}$$



$$\sum_{K_4} C^{[4]}_{4;K4}$$



The simplest example: the gggg amplitude

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which amounts

$$c_{1|2|3|4;\,4} = c^{[0]}_{1|2|3|4;\,4} + c^{[4]}_{1|2|3|4;\,4} = 2i\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}$$

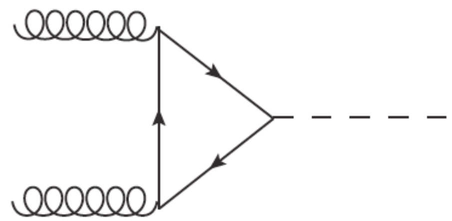
and the full one-loop amplitude

$$\begin{aligned} A_4\left(1_g^+, 2_g^+, 3_g^+, 4_g^+\right) &= c_{1|2|3|4;\,4}\; I_{1|2|3|4}[\mu^4] \\ &= -\frac{i}{48\,\pi^2}\,\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\,, \end{aligned}$$

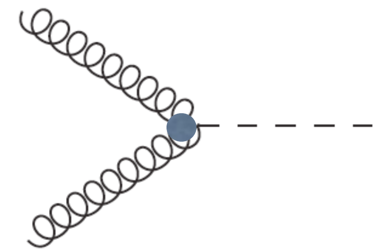
 In agreement with [Bern & Kosower (1992)]

FDF can also be implemented in effective theories

$$gg \to H$$



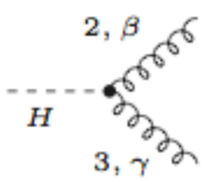
$$\to$$



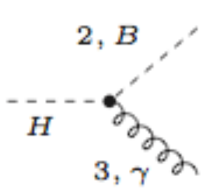
$$\mathcal{L}\,(Hgg) = \frac{\left(\sqrt{2}G_F\right)^{1/2}\alpha_s}{12\pi}\left(1+\frac{11}{4}\frac{\alpha_s}{\pi}\right) H F^a_{\mu\nu}F^{\mu\nu}_a$$

[Adler, Collins and Duncan (1977)]

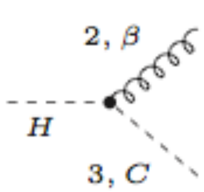
See C. Duhr's Talk



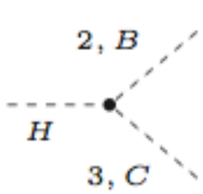
$$= -2i\left[k_3^\beta k_2^\gamma - g^{\beta\gamma}(k_2\cdot k_3+\mu^2)\right],$$



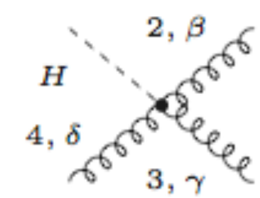
$$= \pm 2\, k_2^\gamma\, \mu\, Q^B\,, \qquad (\tilde{k}_3 = \pm\tilde{\ell}),$$



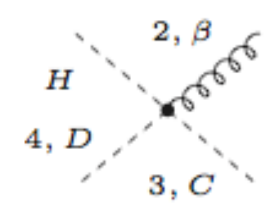
$$= \pm 2\, k_3^\beta\, \mu\, Q^C\,, \qquad (\tilde{k}_2 = \pm\tilde{\ell}),$$



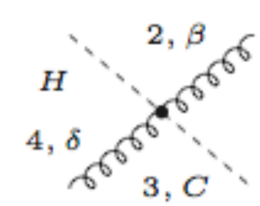
$$= -2i\left[\mu^2\, Q^B Q^C - G^{BC}(k_2\cdot k_3+\mu^2)\right],$$



$$= i\sqrt{2}\left[g_{\beta\gamma}(k_2-k_3)_\delta + g_{\beta\delta}(k_4 - \right.$$
$$\left. + g_{\gamma\delta}(k_3-k_4)_\beta\right],$$

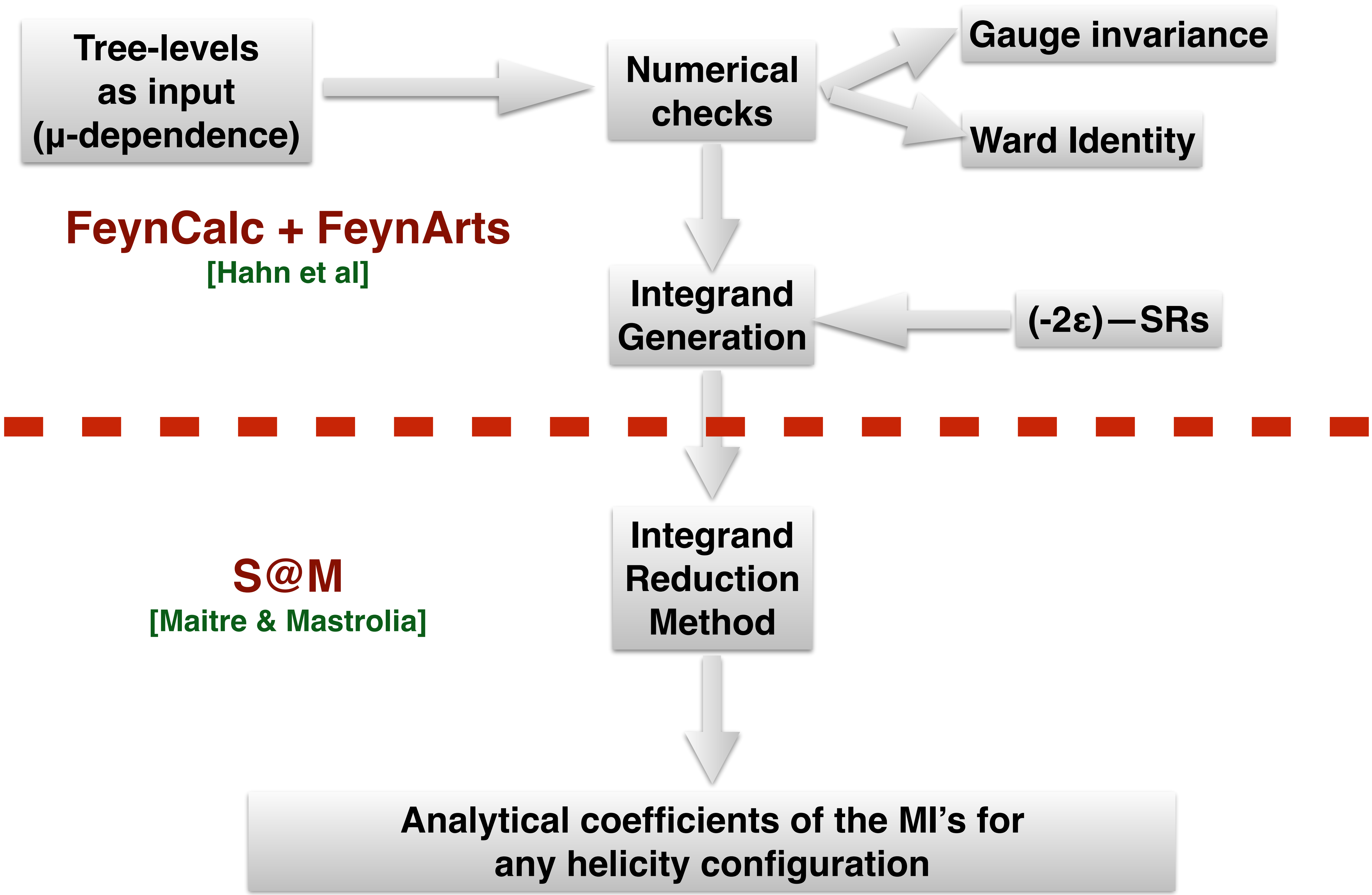


$$= i\sqrt{2}G^{CD}(k_3-k_4)_\beta\,,$$



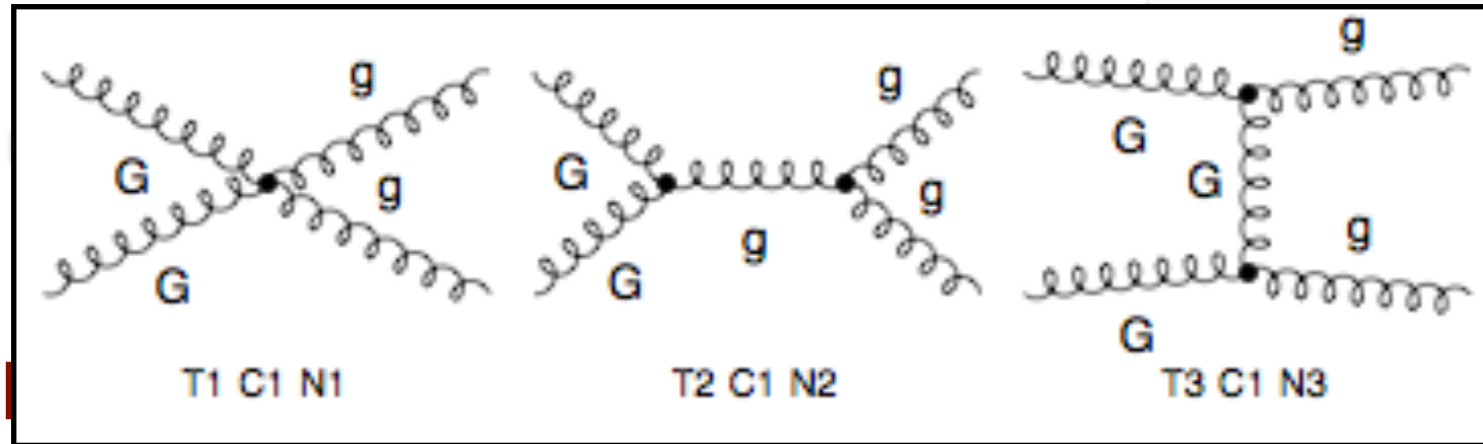
$$= \mp\sqrt{2}g_{\beta\delta}\,\mu\, Q^C\,, \qquad (\tilde{k}_4 - \tilde{k}_2 = \pm\tilde{\ell}).$$

Automation


Tree-levels as input (μ-dependence)
Numerical checks
Gauge invariance
Ward Identity
FeynCalc + FeynArts
[Hahn et al]
Integrand Generation
(-2ε)—SRs
S@M
[Maitre & Mastrolia]
Integrand Reduction Method
Analytical coefficients of the MI's for any helicity configuration

Automation

Tree-levels
as input
(μ-dependence)



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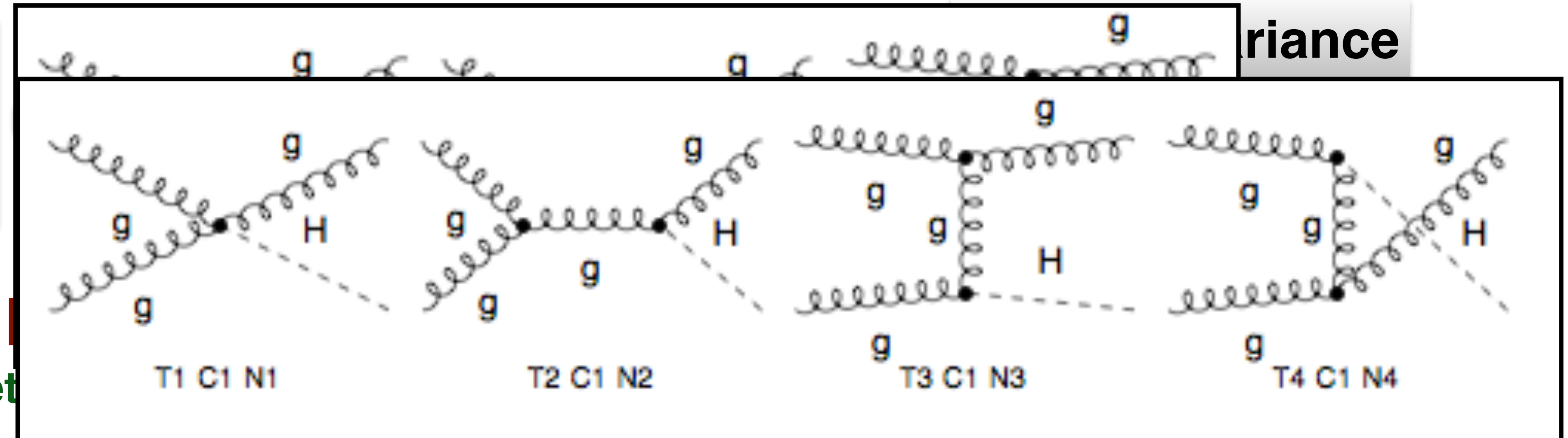

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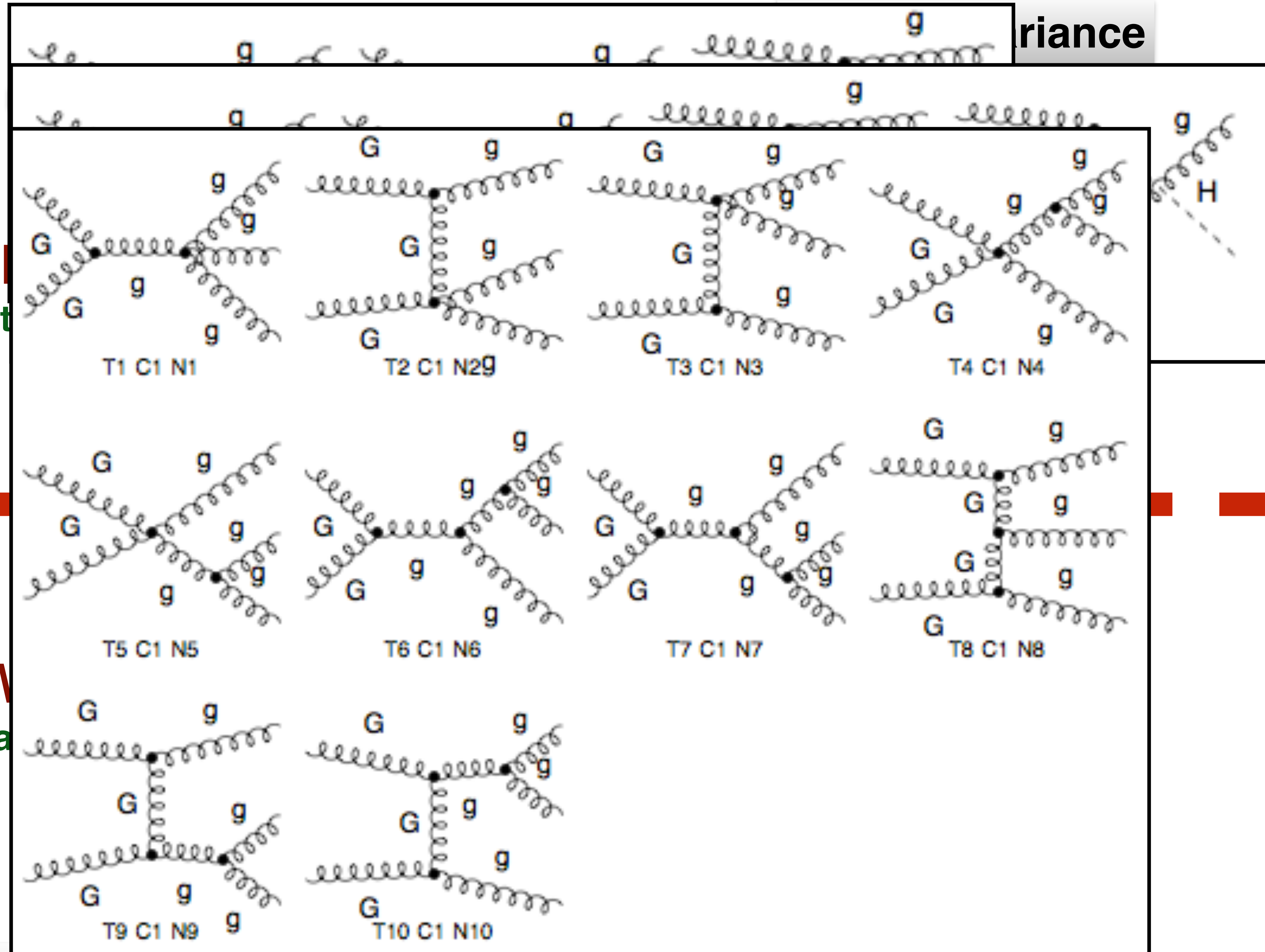

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S@M

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Analytical coefficients of the MI's for any helicity configuration

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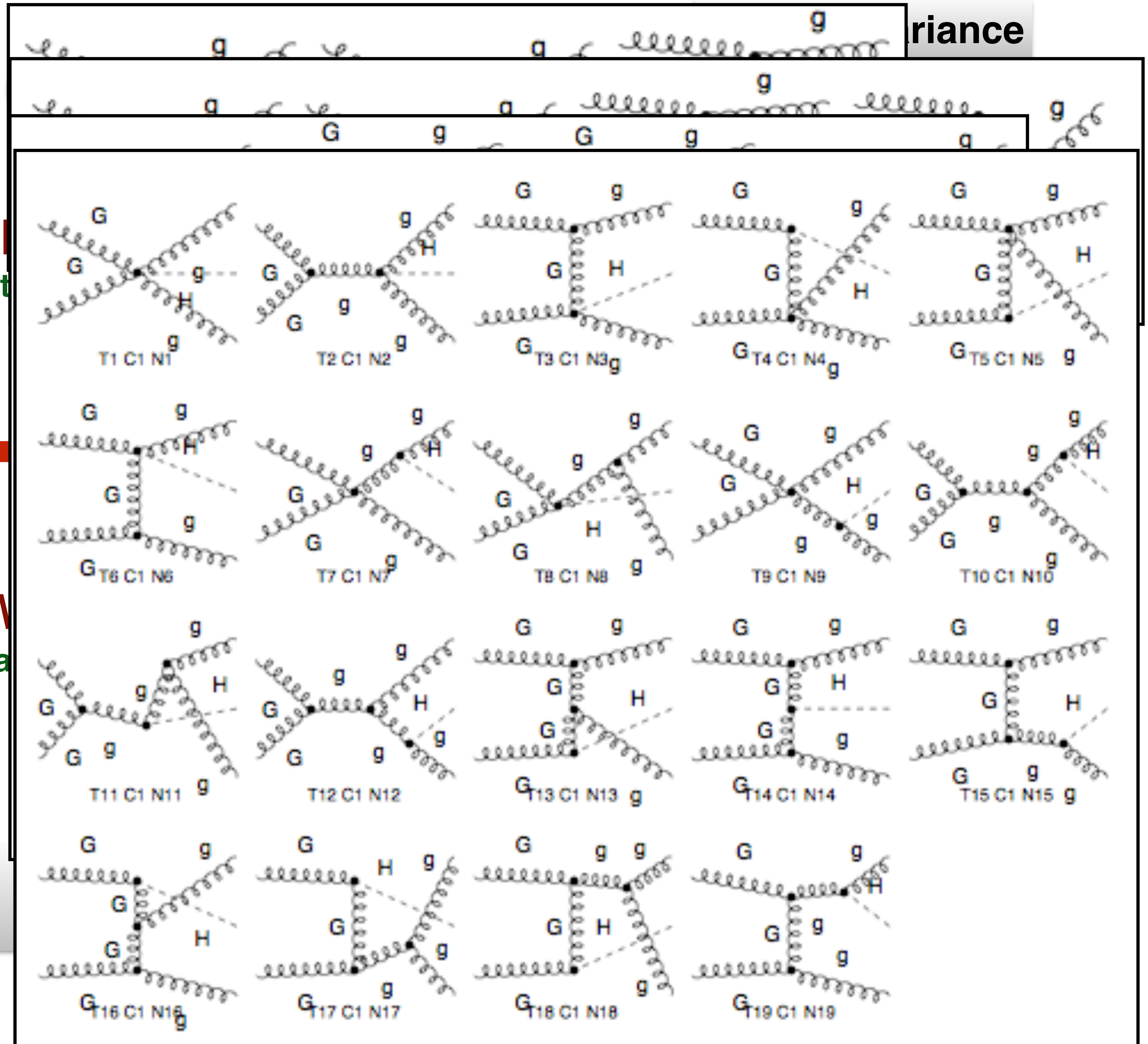
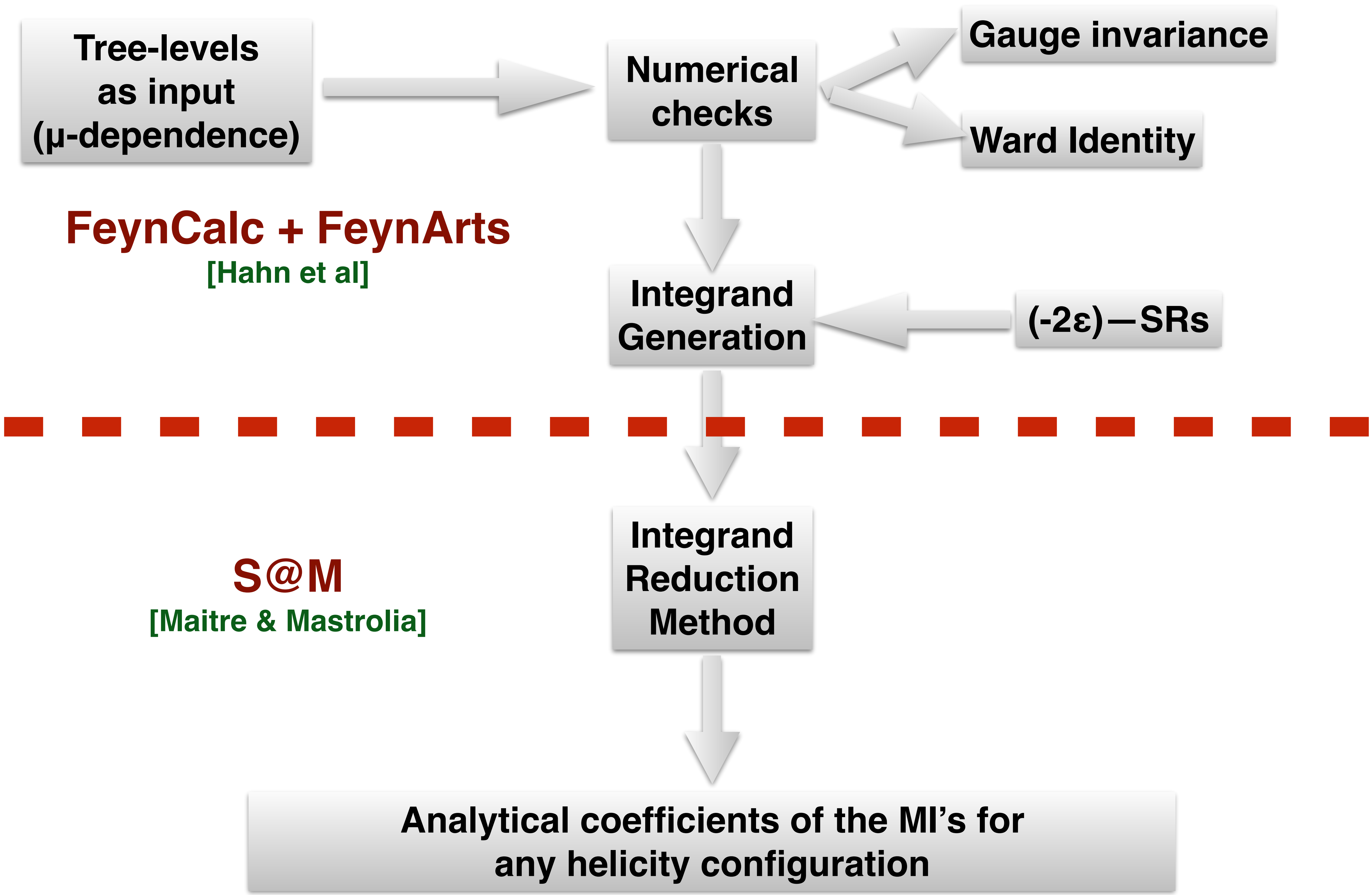


Automation



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Integrand Reduction Method
Analytical coefficients of the MI's for any helicity configuration

Results

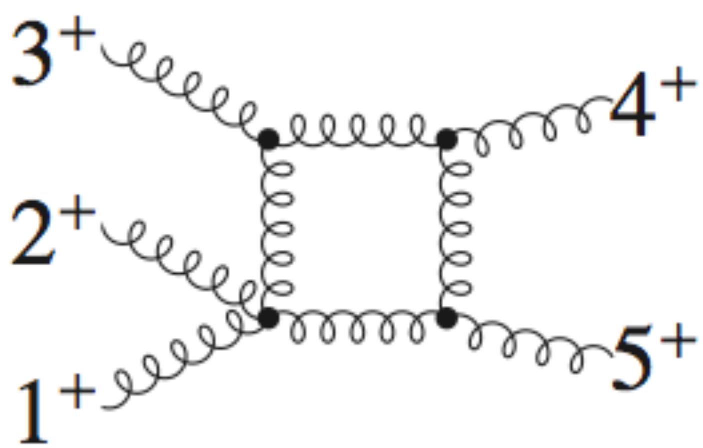
- 4-gluons amplitudes **[Bern and Kosower (1992)]**
- Annihilation of quark & antiquark in two gluons **[Kunszt, Signer and Trocsanyi (1993)]**
- Higgs + 3-gluon amplitudes **[Schmidt (1997)]**
- 5-gluon amplitudes **[Njet]**
- 6-gluon amplitudes **[Njet]**
- Higgs + 4-gluon amplitudes **[done! to be checked with GoSam]**
- Higgs + 5-gluon amplitudes **[in progress, to be checked with GoSam]**

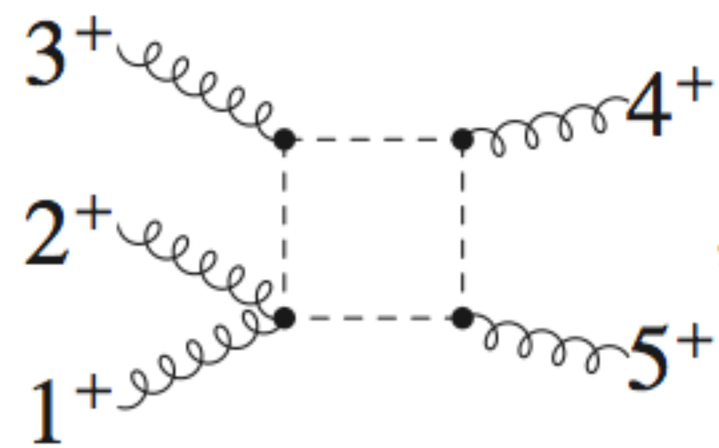
See N. Greiner’s Talk

Results

- 5-gluon amplitudes **[Njet]**

All plus gluon amplitude

$C_{12|3|4|5} =$ 

$+$ 

,

$$c_{12|3|4|5;0} = 0\,,$$

$$c_{12|3|4|5;4} = \frac{2i\,[21]\,[43]\,[53]\,[54]}{\langle 12\rangle\,\mathrm{tr}_5\,(4,1,5,3)}\;;$$

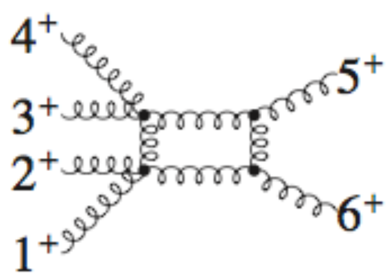
with $\mathrm{tr}_5\,(1,2,3,4) = \langle 1\,|234|\,1] - [1\,|234|\,1\rangle$.

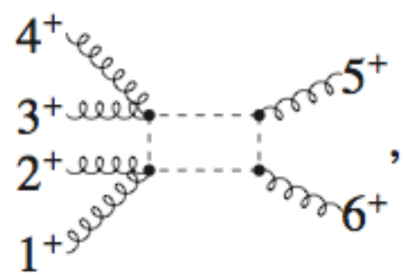
$$\boxed{A_5\left(1^+,2^+,3^+,4^+,5^+\right) = c_{12|3|4|5;4}\, I_{12|3|4|5}\left[\mu^4\right] + \text{cyclic perms.}}$$

Results

- 6-gluon amplitudes **[Njet]**

All plus gluon amplitude

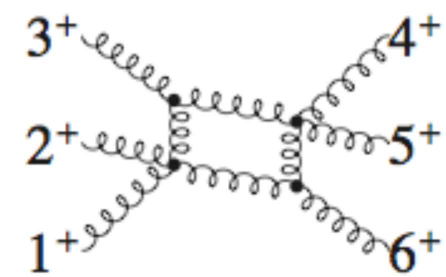
$C_{12|34|5|6} =$ 

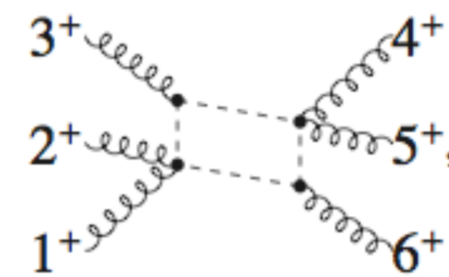
$+$ 

,

$$c_{12|34|5|6;0} = 0\,,$$

$$c_{12|34|5|6;4} = \frac{2i\,\langle 5\,|1+2|\,6]\,\langle 6\,|1+2|\,5]\,[12]\,[43]\,[65]^2}{\langle 12\rangle\,\langle 23\rangle\,\mathrm{tr}_5\,(5,2,6,1)\,\mathrm{tr}_5\,(5,4,6,3)}\,; \tag{32b}$$

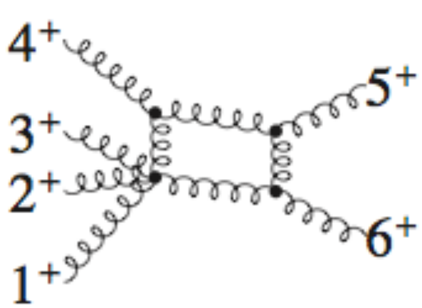
$C_{12|3|45|6} =$ 

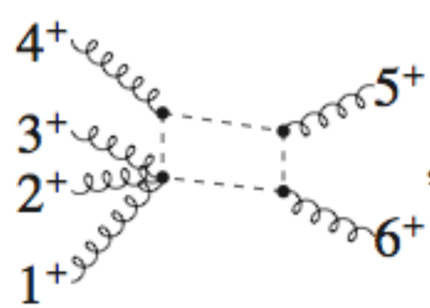
$+$ 

,

$$c_{12|3|45|6;0} = 0\,,$$

$$c_{12|3|45|6;4} = \frac{2i\,[12]\,[54]\,[63]^2}{\langle 12\rangle\,\langle 45\rangle\,\mathrm{tr}_5\,(2,3,6,1)\,\mathrm{tr}_5\,(5,3,6,4)} \times \left(\langle 3\,|1+2|\,3]\,\langle 6\,|1+2|\,6] - s_{36}\,s_{12}\right)\,; \tag{32c}$$

$C_{123|4|5|6} =$ 

$+$ 

,

$$c_{123|4|5|6;0} = 0\,,$$

$$c_{123|4|5|6;4} = \frac{2i\,[56]}{\langle 12\rangle\,\langle 23\rangle\,\mathrm{tr}_5\,(5,4,6,1)\,\mathrm{tr}_5\,(5,4,6,3)} \times \Big(s_{45}\,\langle 6\,|1+2|\,3]\,[51]\,[64]^2 - s_{46}\,\langle 5\,|1+2|\,3]\,[54]^2\,[61]\Big)\,;$$

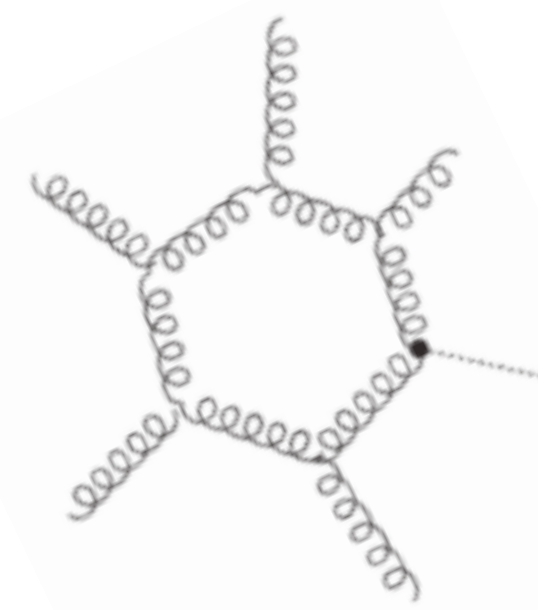
$$\begin{aligned} A_6\,(1^+,2^+,3^+,4^+,5^+,6^+) = {} & c_{123|4|5|6;4}\, I_{123|4|5|6}\left[\mu^4\right] \\ & + c_{12|34|5|6;4}\, I_{12|34|5|6}\left[\mu^4\right] \\ & + \frac{1}{2} c_{12|3|45|6;4}\, I_{12|3|45|6}\left[\mu^4\right] \\ & + \text{cyclic perms.} \end{aligned}$$

Results

- Higgs + 3 gluons **[Mastrolia and WT (to appear)]**

All plus gluon amplitude

at leading order



$$A_6^{\text{tree}}\left(H,1^+,2^+,3^+,4^+,5^+\right)=\frac{i\,m_H^4}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|5\rangle\langle 5|1\rangle}$$

at one-loop

$$\begin{aligned}A_6^{1-\text{loop}}\left(H,1^+,2^+,3^+,4^+,5^+\right)=&\frac{1}{2}A_6^{\text{tree}}\left(s_{1234}s_{1235}-m_H^2 s_{123}\right)I_{123|4|H|5}\left[1\right]\\&+\frac{1}{2}A_6^{\text{tree}}\left(s_{234}s_{345}-s_{34}s_{2345}\right)I_{H1|2|34|5}\left[1\right]\\&-\frac{1}{2}A_6^{\text{tree}}\left(m_H^2-s_{1234}\right)I_{1234|5|H}\left[1\right]\\&+c_{123|4|H|5}I_{123|4|H|5}[\mu^4]+c_{H12|3|4|5}I_{H12|3|4|5}[\mu^4]\\&+c_{1234|5|H}I_{1234|5|H}[\mu^2]+c_{1234|H|5}I_{1234|H|5}[\mu^2]+c_{H123|4|5}I_{H123|4|5}[\mu^2]\\&+c_{H12|34|5}I_{H12|34|5}[\mu^2]+c_{123|4|5H}I_{123|4|5H}[\mu^2]+c_{123|4H|5}I_{123|4H|5}[\mu^2]\\&+c_{12|345H}I_{12|345H}[\mu^2]+c_{123|45H}I_{123|45H}[\mu^2]+c_{H1|2345}I_{H1|2345}[\mu^2]\\&+\text{cyclic perm.}\end{aligned}$$

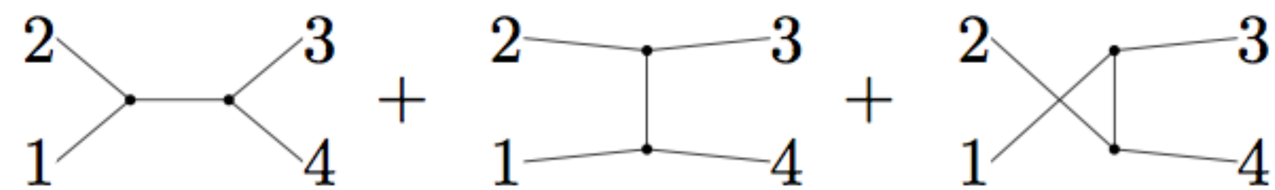
Off-shell colour kinematics duality

[Bern, Carrasco, Johansson (2008),(2010)]
[Bern, Dennen, Huang, Kiermaier (2010)]
[Bern, Carrasco, Dixon, Johansson, Roiban (2012)]
[Du, Luo (2012)]
[Fu, Du, Feng (2012)]
[Nohle (2013)]
[Bern, Davies, Dennen, Huang, Nohle (2015)]
… … …

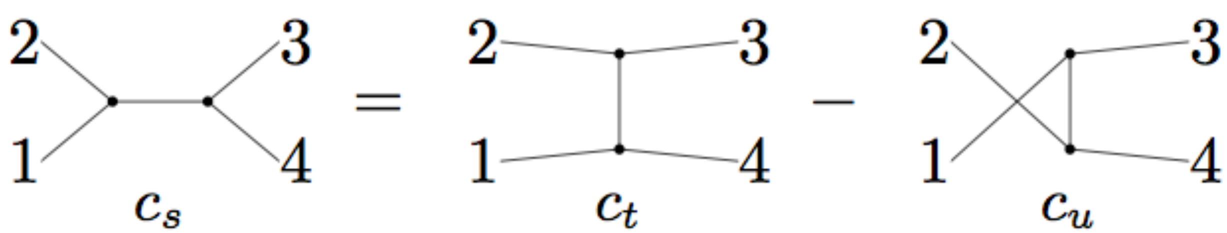
Colour kinematics duality

4-point in gauge theories

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right) =$$



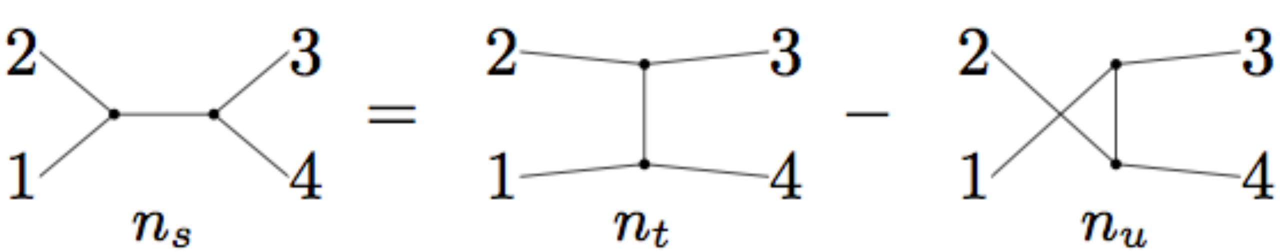
Jacobi Relation (colour)



$$c_s = c_t - c_u$$

$$f^{a_1 a_2 b} f^{a_3 a_4 b} = f^{a_4 a_1 b} f^{a_2 a_3 b} - f^{a_1 a_3 b} f^{a_2 a_4 b}$$

Jacobi identity (Numerators)



$$n_s = n_t - n_u$$

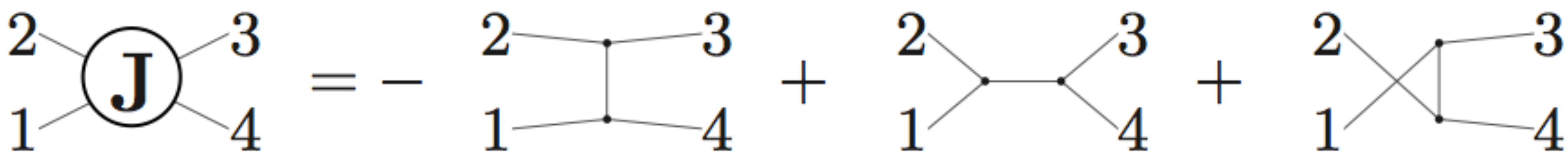
[Bern, Carrasco, Johansson (2008)]

— Satisfied automatically for 4-point tree amplitudes.

— **Conjecture:** For m points and L loops, we can regroup terms between numerators to make this relationship hold.

Off-shell colour kinematics duality

Consider a tensor as the Jacobi identity of numerators



Four-gluon identity $\quad N_g^{\text{tree}} = J^{\mu_1\cdots\mu_4}\varepsilon_{\mu_1}(p_1)\varepsilon_{\mu_2}(p_2)\varepsilon_{\mu_3}(p_3)\varepsilon_{\mu_4}(p_4),$

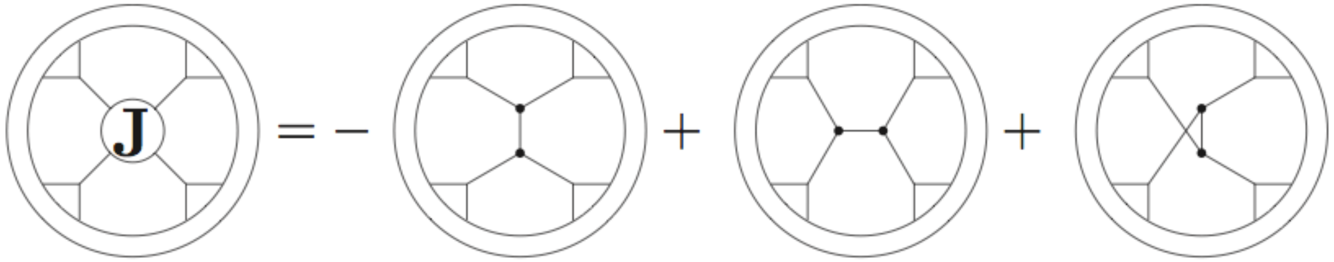
$$
\begin{aligned}
N_g^{\text{tree}} = \varepsilon\left(p_1\right)\cdot p_1 [& \left(\varepsilon\left(p_2\right)\cdot p_1 + 2\varepsilon\left(p_2\right)\cdot p_4\right)\varepsilon\left(p_3\right)\cdot\varepsilon\left(p_4\right) \\
& -\varepsilon\left(p_2\right)\cdot\varepsilon\left(p_4\right)\left(\varepsilon\left(p_3\right)\cdot p_1 + 2\varepsilon\left(p_3\right)\cdot p_4\right) \\
& +\varepsilon\left(p_2\right)\cdot\varepsilon\left(p_3\right)\left(\varepsilon\left(p_4\right)\cdot p_1 + 2\varepsilon\left(p_4\right)\cdot p_3\right)] \\
& \qquad + \text{cyclic permutations.}
\end{aligned}
$$

[Zhu (1980)]

$N_g^{\text{tree}} = 0$ by imposing Momentum Conservation and Transversality condition.

Off-shell colour kinematics duality

At loop level



Outgoing particles in the J-block are now considered as internal:

$$u\left(p_i\right), v\left(p_i\right) \rightarrow \not{p}_i$$
$$\varepsilon^{\mu_i}\left(p_i\,; q_i\right) \rightarrow \Pi^{\mu_i \nu_i}\left(p_i\,; q_i\right)$$

with

$$\Pi^{\mu_i \nu_i}\left(p_i\,; q_i\right) = -g^{\mu_i \nu_i} + \frac{p_i^{\mu_i} q_i^{\nu_i} + p_i^{\nu_i} q_i^{\mu_i}}{p_i \cdot q_i}$$
$$p_{i,\,\mu_i} \Pi^{\mu_i \nu_i}\left(p_i\,; q_i\right) = p_i^2 \frac{q_i^{\nu}}{p_i \cdot q_i}$$
$$\not{p}_i \not{p}_i = p_i^2$$

Numerator built from the J-block decomposed in terms of squared momenta

$$\left(N_{\rm g}^{\rm loop}\right)_{\alpha_1 \ldots \alpha_4} = J^{\mu_1 .. \mu_4} \Pi_{\mu_1 \alpha_1}(p_1, q_1)\, \Pi_{\mu_2 \alpha_2}(p_2, q_2)\, \Pi_{\mu_3 \alpha_3}(p_3, q_3)\, \Pi_{\mu_4 \alpha_4}(p_4, q_4)\,,$$

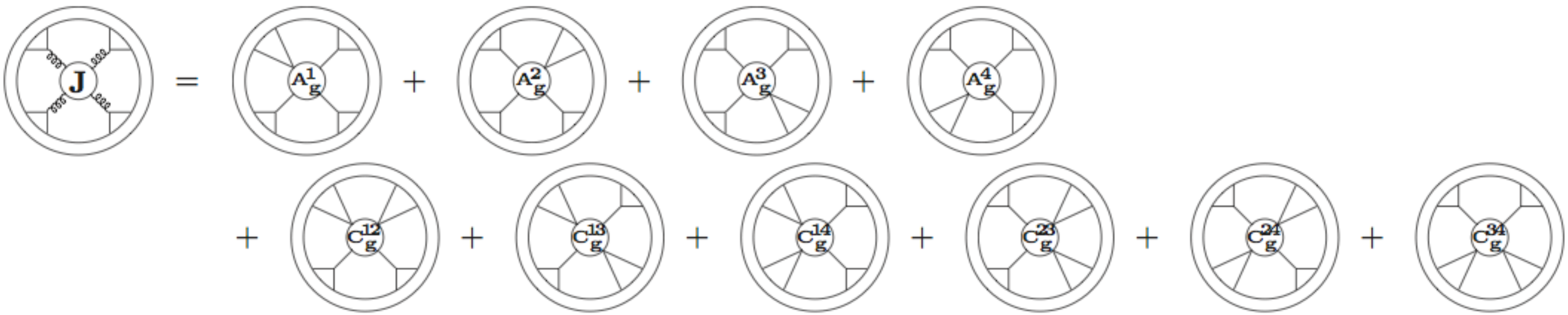
$$\left(N_{\rm g}^{\rm loop}\right)_{\alpha_1 \ldots \alpha_4} = \sum_{i=1}^{4} p_i^2 (A_g^i)_{\alpha_1 \ldots \alpha_4} + \sum_{\substack{i,j=1 \\ i \neq j}}^{4} p_i^2 p_j^2 (C_g^{ij})_{\alpha_1 \ldots \alpha_4}.$$

with
$$A_g = A_g(\{p_i\})$$
$$C_g = C_g(\{p_i\})$$

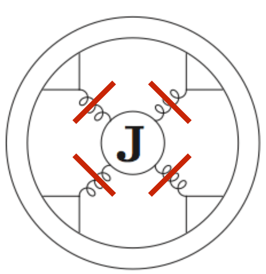
[Mastrolia, Primo, Schubert, WT (to appear)]

Off-shell colour kinematics duality

From the J-block we have



- Any loop diagram built from the *J*-block can be written as the sum of diagrams with one or two propagators less.



$= 0$

- By imposing on-shellness of the four particles

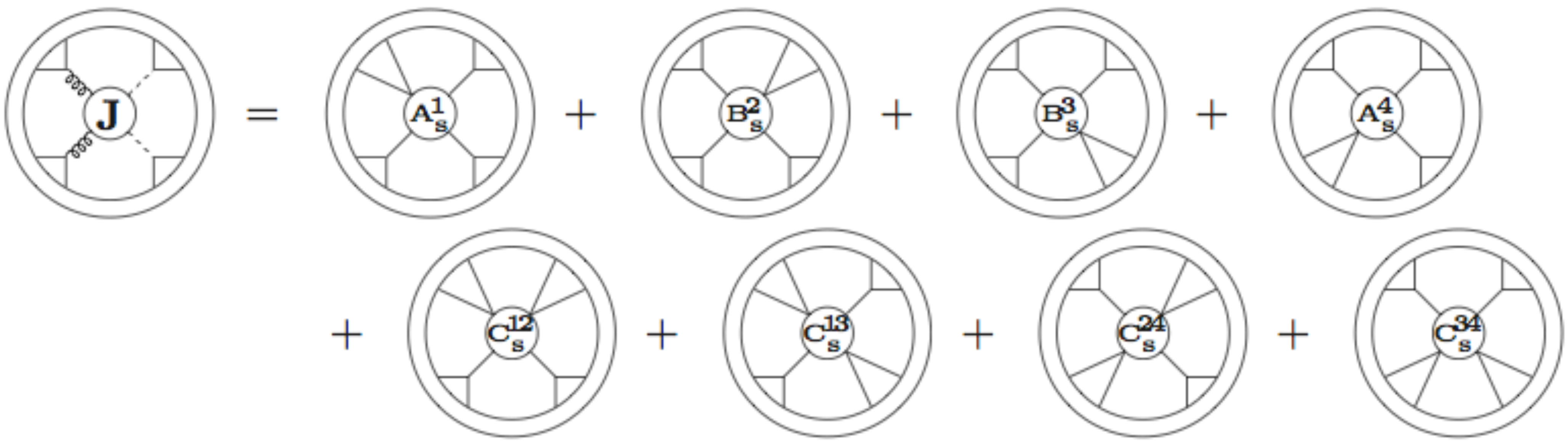
[Mastrolia, Primo, Schubert, WT (to appear)]

Off-shell colour kinematics duality

Gluon and scalars

$$c_s = c_t - c_u$$

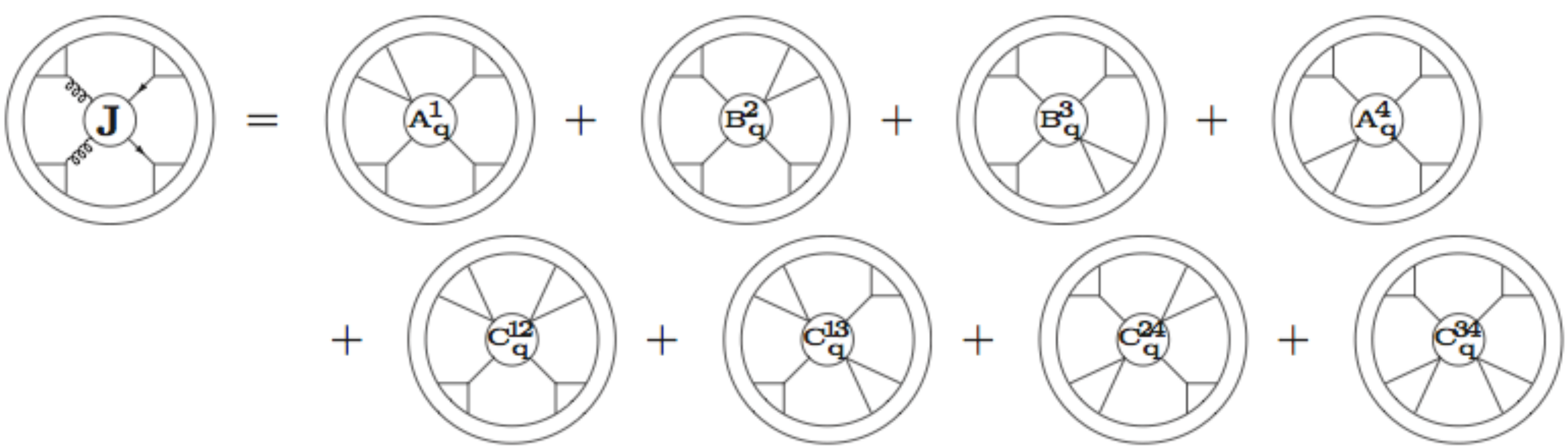
$$f^{a_1 a_2 b} f^{a_3 a_4 b} = f^{a_4 a_1 b} f^{a_2 a_3 b} - f^{a_1 a_3 b} f^{a_2 a_4 b}$$



Gluon and fermions

$$c_s = c_t - c_u$$

$$f^{a_1 a_2 b} T^b_{i_3 \bar{i}_4} = T^{a_1}_{i_3 \bar{k}} T^{a_2}_{k \bar{i}_4} - T^{a_2}_{i_3 \bar{k}} T^{a_1}_{k \bar{i}_4}$$

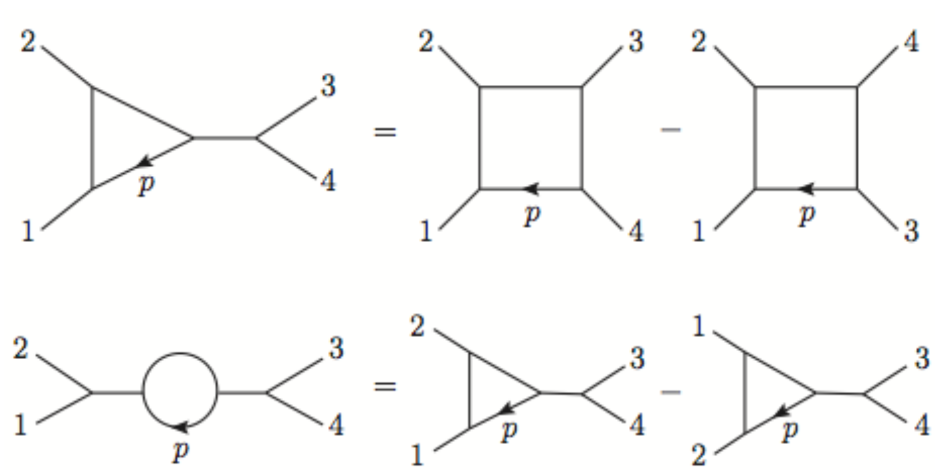


[Mastrolia, Primo, Schubert, WT (to appear)]

Off-shell colour kinematics duality

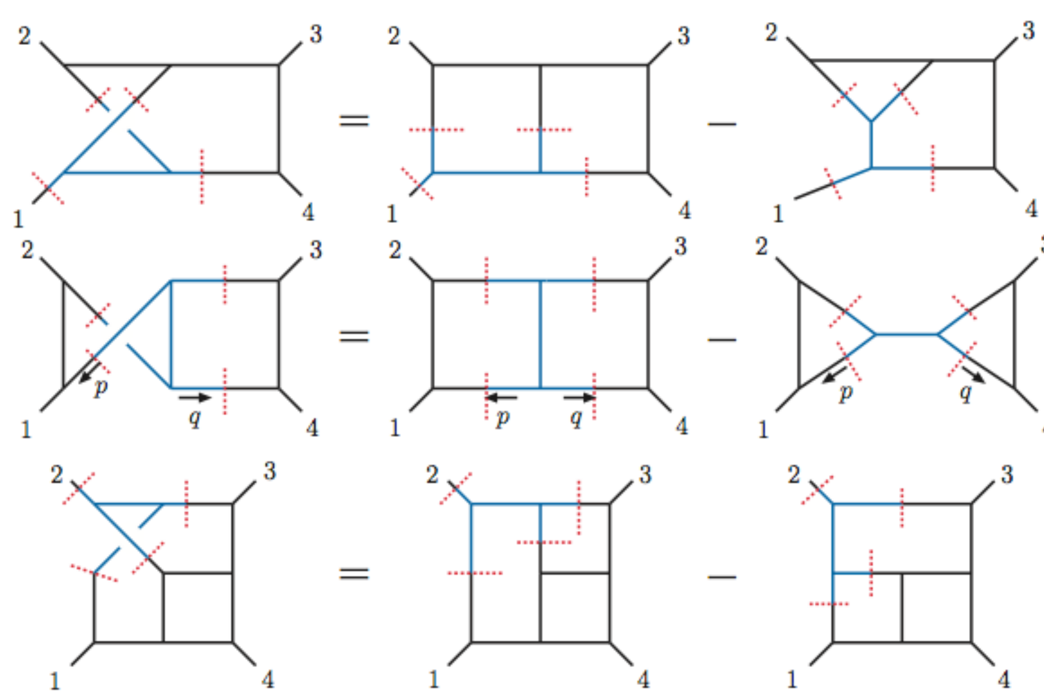
Perspectives: satisfy the colour-kinematics duality at multi-loop level

Dual diagrams at one-loop level



[Nohle (2013)]

Dual diagrams beyond one-loop



[Bern, Carrasco, Johansson (2008)]

Conclusions

PART I

- We introduce the FDF which is a representation of the FDH scheme, where we provide the explicit representation of spinors and polarisation vectors in 4-dimensions
- Any one-loop amplitude is computed at once, without distinguishing between cut constructible and rational part.
- Analytical results of one-loop amplitudes are provided to give one the pieces of the computation of NNLO.
- Perspectives ::: Higgs +3 jets & More loops!

PART 2

- We show explicitly that any loop diagram built from the Jacobi identity (Numerators) can be decomposed in terms of sub-diagrams with one and two propagators less.
- By consequence of the decomposition we trivially proved the colour-kinematics duality is satisfied at multi-loop by only imposing on-shellness of the four particles entering in the "Jacobi"-Block.

Conclusions

PART 1

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PART 2

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- By consequence of the decomposition we trivially proved the colour-kinematics duality is satisfied at multi-loop by only imposing on-shellness of the four particles entering in the "Jacobi"-Block.

Extra Slides

6-dimensional formalism vs FDF

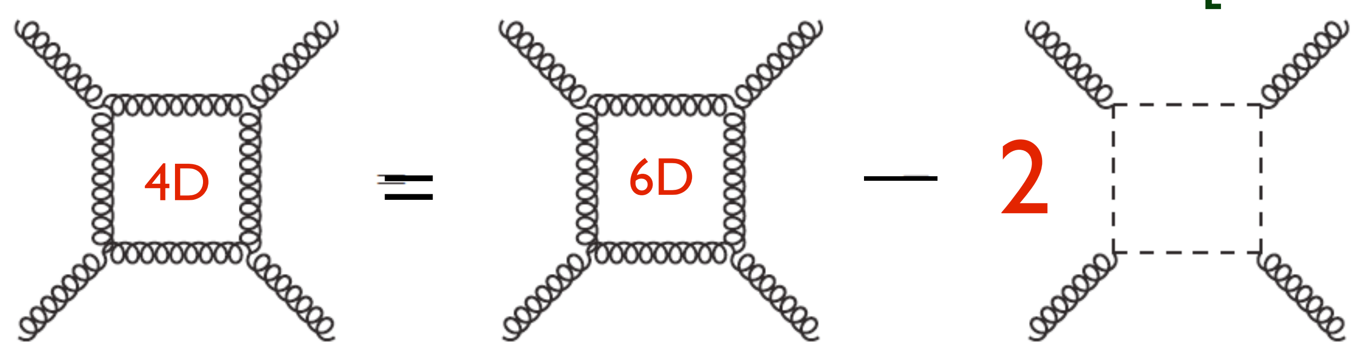
6-dimensional formalism

- D=4 massless gluon: 2 polarisation states
- D=6 massless gluon: 4 polarisation states

Contribution from extra polarisation states must be subtracted out!

4 polarisation states of 6D massless gluon
= 2 polarisation states of 4D massless gluon + two scalars.

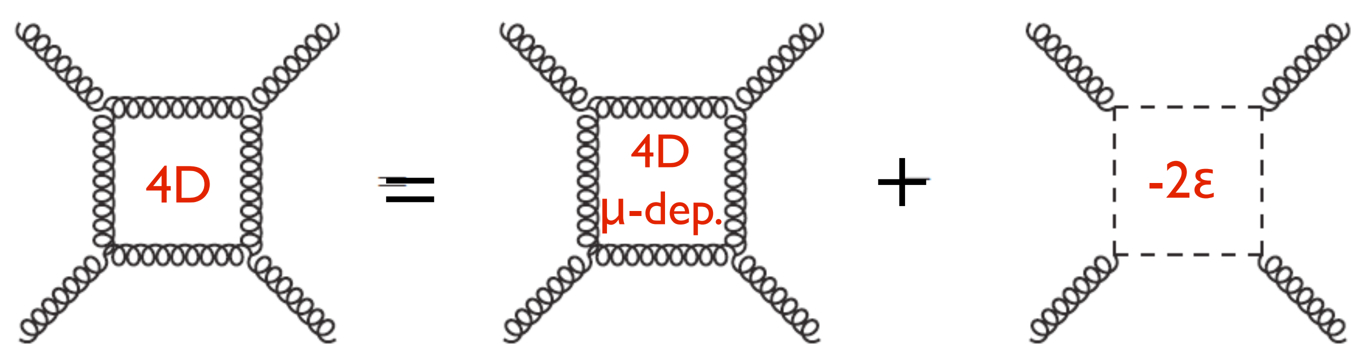
[Giele,Kunszt,Melnikov (2009)]
[Davies (2011)]



FDF

- D=4-2ε massless gluon decomposed into
  3 polarisation states of 4D massive gluon + a -(2ε)-scalar

Contribution come from both massive gluons and (-2ε)-scalar!



[Fazio, Mastrolia, Mirabella, WT (2014)]